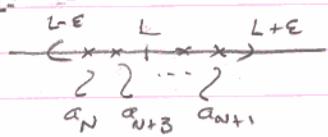
SOME THINGS YOU SHOULD KNOW (HANDOUT 2)

["Def" = Definition, "Thm" = Theorem, "Pf" = Proof, "$\Rightarrow$" = implies]

1. Def: $\lim_{n\to\infty} a_n = L$ means: for all $\varepsilon > 0$, there is an N such that $n \geq N \Rightarrow |a_n - L| < \varepsilon$

TYPICAL PICTURE:



SHORTHAND NOTATION

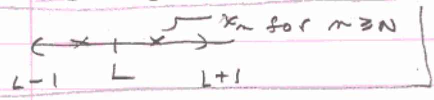
"$a_n \to L$ as $n \to \infty$"

"$a_n \to L$"

SAY "a_n converges"

2. Thm: If a_n converges, then it is bounded above and it is bounded below.

Pf: Say $a_n \to L$. Then from def (let $\varepsilon = 1$) choose N so that $n \geq N \Rightarrow |a_n - L| < 1$. From the picture



$(n \geq N) \Rightarrow a_n \leq L+1$

Let $M = \max\{a_1, a_2, \ldots, a_{N-1}, L+1\}$. Then for all n $a_n \leq M$.

From the picture, $(n \geq N) \Rightarrow a_n \geq L-1$. Let $K = \min\{a_1, \ldots, a_{N-1}, L-1\}$. Then for all n, $a_n \geq K$.

QED

3. $\lim_{n\to\infty} \frac{1}{n} = 0$ Pf: Given $\varepsilon > 0$ choose N so that $N > \frac{1}{\varepsilon}$. Then $\frac{1}{N} < \varepsilon$ and $n \geq N \Rightarrow 0 \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon \Rightarrow |\frac{1}{n} - 0| < \varepsilon$.

4. $0 < r < 1 \Rightarrow \lim r^n = 0$.

Pf: Let $r = \frac{1}{a}$. Then $0 < r < 1 \Rightarrow \frac{1}{a} < 1 \Rightarrow a > 1$.

Let $a = (1 + a_0)$ where $a_0 > 0$. We have

$$0 < r^n = \frac{1}{(1+a_0)^n} = \frac{1}{1 + na_0 + \frac{n(n-1)}{2}a_0^2 + \cdots}$$

$$< \frac{1}{na_0} \quad \text{and} \quad \lim \frac{1}{na_0} = \frac{1}{a_0} \lim \frac{1}{n} = 0$$

Corollary: $|r| < 1 \Rightarrow r^n \to 0$. Pf: $|r^n - 0| = |r^n| = |r|^n$

Def: $a_1+a_2+a_3+\dots = \lim_{N\to\infty} a_1+a_2+\dots+a_N$

ALTERNATIVE NOTATION

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

$$\underbrace{S}_{\text{SUM}} = \lim_{N\to\infty} S_N \qquad S_N = \underbrace{a_1+\dots+a_N}_{\text{partial sum}}$$

Thm: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

Pf: $a_N = (a_1+\dots+a_{N-1}+a_N) - (a_1+\dots+a_{N-1})$

$= S_N - S_{N-1}$

and $\left.\begin{array}{l} S_N \to S \\ S_{N-1}\to S \end{array}\right\} \Rightarrow \lim_{N\to\infty} a_N = \lim_{N\to\infty} (S_N - S_{N-1})$

$= S - S = 0.$

CONVERSE IS FALSE

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots \qquad a_n = \frac{1}{n} \to 0$ "HARMONIC SERIES"

BUT $\sum_{n=1}^{\infty} \frac{1}{n}$ DOES NOT CONVERGE

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\dots$$

$$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\dots$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots = \infty \qquad (\text{even } a_n \to 0)$$

Lemma: $\frac{1-r^{n+1}}{1-r} = 1+r+r^2+\dots+r^n$

Pf: $(1-r)(1+r+r^2+\dots+r^n)$

$= (1+r+\dots+r^n) - (r+r^2+\dots+r^n+r^{n+1})$

$= 1-r^{n+1}$

Th: $|r|<1 \Rightarrow a+ar+ar^2+\dots = \frac{a}{1-r}$ (geometric series)

$S_N = a+\dots+ar^N \quad \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r} = \frac{a}{1-r}(1$