Answers to practice exam #2.

1. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\dots$

$$
\begin{array}{rl}
 & 1 + \frac{x^2}{2} - \frac{x^3}{3} - \frac{7}{24}x^4 \\
1+x \; \big) & 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\dots \\
 & \underline{1+x} \\
 & \frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\dots \\
 & \underline{\frac{x^2}{2}+\frac{x^3}{2}} \\
 & -\frac{x^3}{3}+\frac{x^4}{24}+\dots \\
 & \underline{-\frac{x^3}{3}-\frac{x^4}{3}} \\
 & -\frac{7}{24}x^4+\dots \\
 & \underline{-\frac{7}{24}x^4-\frac{7}{24}x^5}
\end{array}
$$

$-\frac{1}{3} = \frac{1-8}{24} = -\frac{7}{24}$

OR MULTIPLY power series for e^x and for $\frac{1}{1+x}$

$\to 1-x+x^2-x^3+\dots$

$(a=1, r=-x)$

2. $\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt$

$a=1, r=-t^2$

$\frac{1}{1+t^2} = 1-t^2+t^4-t^6+\dots$

$= \int_0^x \left[1-t^2+t^4-t^6+\dots\right] dt$

$= x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$

3.

$f(x)=e^x$	$f(1)=e$	$a_0=e$
$f'(x)=e^x$	$f'(1)=e$	$a_1=\frac{e}{1!}$
$\vdots$	$\vdots$	$a_2=\frac{e}{2!}$
		$a_3=\frac{e}{3!}$

Power Series

$$e+e(x-1)+\frac{e}{2!}(x-1)^2+\frac{e}{3!}(x-1)^3+\dots+\frac{e}{n!}(x-1)^n+R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-1)^{n+1} = \frac{e^z}{(n+1)!}(x-1)^{n+1}$$

$$\boxed{\begin{array}{l} 1<z<x \Rightarrow e^z<e^x \\ \text{or } x<z<1 \Rightarrow e^z<e^1 \end{array}}$$

$$|R_n(x)| = e^{\xi}\frac{|x-1|^{n+1}}{(n+1)!} < C\frac{|x-1|^{n+1}}{(n+1)!} \qquad C = \max\{e^x, e^1\}$$

$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} C\frac{|x-1|^{n+1}}{(n+1)!} = 0$ because in general

$a > 0 \Rightarrow \lim \frac{a^{n+1}}{(n+1)!} = \lim \frac{a}{1}\cdots\frac{a}{N+1} = 0$

$\to$ why? see $\to$

4. a) Let $r < r_1 < 1$. Then $\left|\frac{a_{n+1}}{a_n}\right| \to r$ implies that there is an N such that $n \ge N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < r_1$. $\quad \varepsilon = r_1 - r$

It follows that $|a_{N+1}| < r_1|a_N|$, $|a_{N+2}| < r_1|a_{N+1}| < r_1^2|a_N|$, ...

$$|a_1| + |a_2| + \dots + |a_{N-1}| + |a_N| + |a_{N+1}| + |a_{N+2}| + \dots$$

$$< |a_1| + |a_2| + \dots + |a_N|\,[\underbrace{1 + r_1 + r_1^2 + \dots}_{\frac{1}{1-r_1}}] < \infty$$

Thus $\sum_{n=1}^{\infty} |a_n|$ converges and by absolute convergence test, $\sum_{n=1}^{\infty} a_n$ converges

b) We apply the ratio test (i.e. a)) to the series $\sum a_n x^n$

$$\lim \left|\frac{(n+1)\,a_{n+1}\,x^n}{n\,a_n\,x^{n-1}}\right| = \lim (1+\tfrac{1}{n})\left|\frac{a_{n+1}}{a_n}\right| |x| = r|x|$$

We have $r|x| < 1 \iff |x| < \frac{1}{r}$.

5 a)  $f_n(x) = x^n \quad f(x) = \begin{cases} 0 & x<1 \\ 1 & x=1 \end{cases} \quad I = [0,1]$

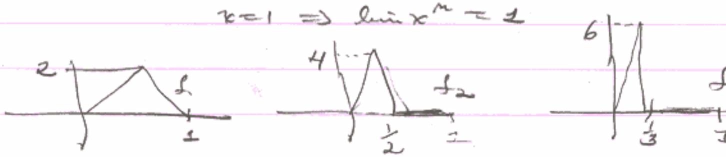
$x < 1 \Rightarrow \lim x^n = 0$

$x = 1 \Rightarrow \lim x^n = 1$

$\|f_n - f\|_\infty = 1$

$\|f_n - f\|_\infty \not\to 0$

$I = [0,1]$

b) 

$\|f_n - f\| = n$

$\|f_n - f\|_\infty \not\to 0$

Let $f(x) = 0$ for all x. Then $f_n(x) \to f(x)$ for all $x \in I$

On the other hand, $\int_0^1 f_n(x)\,dx = 1$ area under curves

$\int_0^1 f(x)\,dx = 0$ so $\int_0^1 f_n(x)\,dx \not\to \int_0^1 f(x)\,dx$.