Printed Name WARNING!! WE ARE XEROXING THE EXAMS.

Signature ______ ID Number ______ Sec. ______

Math 33b/2 Hour Exam #1 4/26/02. **This is a closed book exam, and calculators are not to be used**. Note: everyone gets 10 free points.

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1(20). Find the following limits (include calculations!):

a) $\lim_{x\to 0} \frac{e^{3x}-1}{1-\cos x} \underset{(\frac{0}{0})}{=} \lim_{x\to 0} \frac{3e^{3x}}{\sin x} = \frac{3e^0}{0} = \frac{3}{0} = \infty$

b) $\lim_{x\to \pi/2} \frac{\sin x}{x} = \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{0}{\frac{\pi}{2}} = \frac{2}{\pi}$

c) $\lim_{n\to\infty} \frac{n!}{5^n} = \lim_{n\to\infty} \frac{1}{5}\cdot\frac{2}{5}\cdot\ldots\cdot\frac{9}{5}\cdot\frac{10}{5}\cdot\frac{11}{5}\cdot\ldots$

$\geq \lim_{n\to\infty} \frac{9!}{5^9} 2^{n-9} = \frac{9!}{5!} 2^{\infty} = \infty$

NOTE: you had to include a calculation similar to this to get full credit.

d) $\lim_{x\to \cancel{0+}\,\infty} x^{\frac{1}{x}}$

Let $L = \lim_{x\to\infty} x^{\frac{1}{x}}$. Then

$\ln L = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

$L = e^{\ln L} = e^0 = 1.$

2(20). Determine which of the following sums converge. Carefully explain which test you are using.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ (limit comparison test) use $\sum \frac{1}{n^2} < \infty$ (p-test, $p=2$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-n+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-n+1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}+\frac{1}{n^2}}$$

$$= 1 < \infty$$

hence $\sum \frac{1}{n^2} < \infty \Rightarrow \sum \frac{1}{n^2-n+1} < \infty$

b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (integral test)

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{R\to\infty} \int_2^R \frac{dx}{x(\ln x)^2}$$

$$= \lim_{R\to\infty} \left[-\frac{1}{\ln x} \right]_2^R$$

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -u^{-1} = -\frac{1}{\ln x}$$

$$= \lim_{R\to\infty} \left[\frac{1}{\ln 2} - \frac{1}{\ln R} \right]$$

$(u = \ln x)$

$du = \frac{dx}{x}$ → (YOU CAN USE DERIVATIVE TEST)

$$= \frac{1}{\ln 2} < \infty$$

SINCE $f(x) = \frac{1}{x(\ln x)^2}$ is decreasing, $\sum_2^{\infty} f(n) < \infty$.

c) $1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \cdots$ [be careful!]

NOTE: This is an example of a sum $\sum_{n=0}^{\infty} (-1)^n a_n$ where $a_n \geq 0$ and $a_n \to 0$.

However you do not have $a_0 \geq a_1 \geq a_2 \geq \ldots$ so you can't use the alternating series test. On the other hand note that

$$\left(1-\frac{1}{2}\right) + \left(\frac{2}{3}-\frac{1}{3}\right) + \left(\frac{2}{4}-\frac{1}{4}\right) + \ldots = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

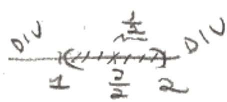
which diverges

d) Determine where the series $\sum_{n=1}^{\infty} (-1)^n \frac{(2x-3)^n}{n}$ converges (include a careful picture of the points x for which it converges).

$$\rho = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \frac{\frac{(2x-3)^{n+1}}{n+1}}{\frac{(2x-3)^n}{n}} = \lim_{n\to\infty} \frac{n}{n+1} |2x-3| = |2x-3|$$

Convergence if $\rho = |2x-3| < 1$

$|2x-3| < 1 \Leftrightarrow |x - \frac{3}{2}| < \frac{1}{2}$



DIVERGENCE $|2x-3| > 1 \Leftrightarrow |x-\frac{3}{2}| > \frac{1}{2}$

$x=1$: $\sum_{n=1}^{\infty} \frac{(-1)^n (2-3)^n}{n} = \sum \frac{1}{n} = \infty$ DIVERGENCE (p-test, $p=1$)

$x=2$: $\sum_{n=1} \frac{(-1)^n (4-3)^n}{n} = \sum \frac{(-1)^n}{n}$ CONVERGENCE (Alt. series)

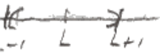
3 a)(5) **Define** $\lim_{n\to\infty} x_n = L$. For all $\varepsilon > 0$, there exists an N such that

$n \geq N \Rightarrow |x_n - L| < \varepsilon$

b)(10) **Prove** that if $\lim_{n\to\infty} x_n = L$, then x_n is bounded below (include a picture!).

Choose N such that $n \geq N \Rightarrow |x_n - L| < 1$.

$\Rightarrow x_n > L - 1$



Let $K = \min\{x_1, \dots, x_{N-1}, L-1\}$. Then $x_n \geq K$ for all n.

c)(5) **Define**: the series $\sum_{n=1}^{\infty} a_n$ converges.

The sequence of partial sums $S_N = \sum_{n=1}^{N} a_n$ converges.

d)(10) **Prove** that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum a_n$ converges.

$|a_n| = a_n^+ + a_n^- \Rightarrow a_n^+ \leq |a_n|$

$a_n^- \leq |a_n|$

$0 \leq \sum_{n=1}^{\infty} a_n^+ \leq \sum_{n=1}^{\infty} |a_n| \Rightarrow \sum_{n=1}^{\infty} a_n^+$ converges

$0 \leq \sum_{n=1}^{\infty} a_n^- \leq \sum |a_n| \Rightarrow \sum_{n=1}^{\infty} a_n^-$ converges

} comparison test

Since $\sum_1^N a_n = \sum_1^N (a_n^+ - a_n^-) = \sum_1^N a_n^+ - \sum_1^N a_n^-$,

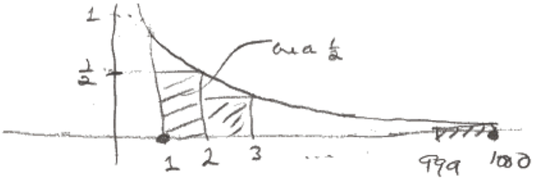
it follows that $\lim_{N\to\infty} \sum_1^N a_n = \lim_{N\to\infty} \sum_1^N a_n^+ - \lim_{N\to\infty} \sum_1^N a_n^-$

$= \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$ converges

4.(20) Find a constant C such that

$$\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{1000}<C.$$

Hint: As in the proof of the integral test, draw the graph of $\frac{1}{x}$ $(1 \le x \le 1000)$ and consider appropriate rectangles under it.



$$0 = \underbrace{\int_1^{1000} \frac{dx}{x}}_{\substack{\text{area under}\\ \text{curve between}\\ \text{1 and 1000}}} > \underbrace{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000}}_{\text{shaded area}}$$

<u>NOTE</u>: If you didn't <u>carefully</u> draw the rectangles you were apt to get errors, such as $\int_2^{\infty}\frac{dx}{x}$, and you would lose $\sim\frac{1}{2}$ credit.