

Printed Name _____

Signature _____ ID Number _____ Sec. _____

Math 33b/2 Hour Exam #2 5/17/02. This is a closed book exam, and calculators are not to be used.

1. Find the power series for $\sec x = \frac{1}{\cos x}$ out to the x^4 term. Suggestion: use division of power series.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \quad \begin{array}{r} 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots \\ \hline 1 - \frac{x^2}{2} + \frac{x^4}{24} \end{array}$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$\frac{x^2}{2} - \frac{x^4}{24} + \dots$$

$$\frac{x^2}{2} - \frac{x^4}{24} \dots$$

$$\frac{5}{24}x^4 + \dots$$

$$\frac{5}{24}x^4 + \dots$$

$$0 + \dots$$

1
2
3
4
5
T

$$-\frac{1}{24} + \frac{1}{24}$$

$$= \frac{-1+1}{24} = \frac{0}{24}$$

2. Using power series, find the coefficients a, b, c, d in the following expression

$$2x^3 + 3x^2 - 2x + 1 = a(x+1)^3 + b(x+1)^2 + c(x+1) + d = \sum a_n (x+1)^n$$

$$a_n = f^{(n)}(-1)/n!$$

$$f(x) = 2x^3 + 3x^2 - 2x + 1$$

$$f(-1) = -2 + 3 = 1$$

$$a_0 = \frac{1}{0!} = 1$$

$$f'(x) = 6x^2 + 6x - 2$$

$$f'(-1) = 6 - 6 - 2 = -2$$

$$a_1 = \frac{-2}{1!} = -2$$

$$f''(x) = 12x + 6$$

$$f''(-1) = -12 + 6 = -6$$

$$a_2 = \frac{-6}{2!} = -3$$

$$f^{(3)}(x) = 12$$

$$f^{(3)}(-1) = 12$$

$$a_3 = \frac{12}{3!} = 2$$

$$f^{(4)}(x) = 0$$

$$f^{(4)}(-1) = \dots = 0$$

$$\rightarrow = 1(x+1)^0 + (-2)(x+1)^1 + (-3)(x+1)^2 + 2(x+1)^3$$

$$\uparrow$$

d

$$\uparrow$$

c

$$\uparrow$$

b

$$\uparrow$$

a

3. Use the remainder term $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$ to prove that if $-\frac{1}{2} \leq x < 1$, then the power series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

converges to $f(x) = \ln(1+x)$. You may use the fact that for any k , $f^{(k)}(x) = (-1)^{k-1} (k-1)! (1+x)^{-k}$. Hint: remember that ξ lies between 0 and x .

$$f^{(n+1)}(\xi) = (-1)^n \cdot n! \cdot (1+\xi)^{-n-1}$$

$$|R_n(x)| = \left| \frac{(-1)^n n! (1+\xi)^{-n-1}}{(n+1)!} x^{n+1} \right| = \frac{1}{n+1} \cdot \frac{1}{(1+\xi)^{n+1}} |x|^{n+1}$$

By $0 \leq x < 1$, then since $0 < \xi < x$, $\frac{1}{(1+\xi)^{n+1}} \leq \frac{1}{1}$ and

$$|R_n(x)| \leq \frac{|x|^{n+1}}{n+1} < \frac{1}{n} \rightarrow 0$$

4. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then a_n converges to zero.

By definition, the sequence of partial sums S_N converges to $S = \sum_{n=1}^{\infty} a_n$.

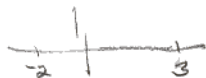
We have $S_N \rightarrow S$ implies $S_{N-1} \rightarrow S$ as $N \rightarrow \infty$ and thus

$$a_n = S_N - S_{N-1} \rightarrow S - S = 0$$

5. Suppose that $f(x)$ is defined on an interval I .

a) Define: $\|f\|_{\infty} = \sup\{|f(x)| : x \in I\}$

b) Let $I = [-2, 2]$ and $f = \frac{1}{3}x^3 - 3x^2 + 8x$. Compute $\|f\|_{\infty}$.



$$f'(x) = x^2 - 6x + 8 = 0 \Rightarrow (x-2)(x-4) = 0$$

$$\Rightarrow x=2, x=4$$

NOT IN INTERVAL

x	f(x)
-2	$-8/3 - 12 - 16 = -92/3$
3	$9 - 27 + 24 = 6$
2	$8/3 - 12 + 16 = 20/3$

$$-28 - \frac{2}{3} = -\frac{84}{3} - \frac{2}{3} = -\frac{86}{3}$$

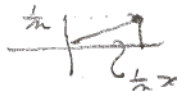
$$\|f\|_{\infty} = 92 - 2 \frac{2}{3}$$

c) Use $\| \cdot \|_{\infty}$ to define: the sequence of functions f_n on I converges uniformly to f .

$$\|f_n - f\|_{\infty} \rightarrow 0$$

d) Let $I = (0, 1]$. Does the sequence $f_n(x) = (1 + \frac{1}{n})x$ converge uniformly to $f(x) = x$?
Prove your assertion.

$$\|f_n - f\|_{\infty} = \|(1 + \frac{1}{n})x - x\|_{\infty} = \|\frac{1}{n}x\|_{\infty} = \frac{1}{n}$$



yes

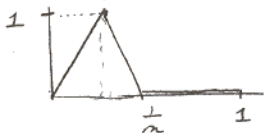
e) Let $I = (0, 1]$. Does the sequence $f_n(x) = (1 + \frac{1}{n})(\frac{1}{x})$ converge uniformly to $f(x) = \frac{1}{x}$?
Prove your assertion.

$$\|f_n - f\|_{\infty} = \|(1 + \frac{1}{n}) \cdot \frac{1}{x} - \frac{1}{x}\|_{\infty} = \infty$$



NO since $\|f_n - f\|_{\infty} \rightarrow \infty$

f) Let $I = [0, 1]$ and define $f_n(x)$ as in the diagram below, and $f(x) = 0$.



i) Does $\int_0^1 f_n(x) dx$ converge to $\int_0^1 f(x) dx$? Prove your assertion.

$$\int_0^1 f_n(x) dx = \text{area of triangle} = \frac{1}{2} \cdot \frac{1}{n} \cdot 1 = \frac{1}{2n} \rightarrow 0 = \int_0^1 f(x) dx$$

yes

ii) Does f_n converge uniformly to f ? Prove your assertion.

$$\|f_n - f\|_{\infty} = \|f_n\|_{\infty} = 1 \text{ since } f_n(\frac{1}{2n}) = 1.$$

hence $\|f_n - f\|_{\infty} \not\rightarrow 0$ i.e., f_n does not converge uniformly.