b) $(\|f\|_2)^2 = f\odot f = \left(\sum_n c_n e^{inx}\right)\odot\left(\sum c_m e^{imx}\right)$

$$= \sum_{n,m}\left(c_n e^{inx}\odot c_m e^{imx}\right)$$

$$= \sum_n c_n e^{inx}\odot c_n e^{inx} \qquad \left(e^{inx}\odot e^{imx}=0 \text{ if } n\neq m\right)$$

$e^{inx}\perp e^{imx}$ i.e.

$$= \sum_n (c_n\bar{c}_n)\, e^{inx}\odot e^{inx}$$

$$= \sum_n |c_n|^2 \left(\|e^{inx}\|_2\right)^2 = \sum_n |c_n|^2\, 2\pi$$

$$\Rightarrow \|f\|_2 = \sqrt{2\pi}\left(\sum |c_n|^2\right)^{\frac{1}{2}}.$$

c) We have that if $f(x)=x$, then from problem 2

$$\|f\|_2 = \sqrt{\frac{2}{3}}\,\pi^{3/2}$$

On the other hand

$$f(x) = x = \sum_{\substack{-\infty\\ n\neq 0}}^{\infty} \overbrace{\frac{i}{n}(-1)^n}^{c_n} e^{inx} \qquad \text{(see page 388)}$$

and thus $(\|f\|_2)^2 = 2\pi \sum |c_n|^2$

$$= 2\pi \sum_{n\neq 0} \left|\frac{i}{n}(-1)^n e^{inx}\right|^2 = 2\pi \sum_{n\neq 0}\frac{1}{n^2}$$

$$= 2\pi\cdot 2\sum_{n>0}\frac{1}{n^2}$$

Hence we have $\frac{2}{3}\pi^3 = 4\pi\sum_{n>0}\frac{1}{n^2} \Rightarrow \sum_{n>0}\frac{1}{n^2} = \frac{\pi^2}{6}.$

p. 388: 1. $f(x) = x^2$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}dx$$

$$= \underbrace{\frac{1}{2\pi}\left[x^2\frac{e^{-inx}}{(-in)}\right]_{-\pi}^{\pi}}_{=0 \text{ since } x^2 \text{ and } e^{-inx} \text{ are equal at } \pi \text{ and } (-\pi)} - \frac{1}{2\pi}\int_{-\pi}^{\pi} 2x\frac{e^{-inx}}{(-in)}dx$$

Assignment 9

1. $f(x) = x \quad g(x) = \sin x$

$$f \odot g = \int_{-\pi}^{\pi} x \sin x \, dx = \left[x(-\cos x) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (-\cos x) dx$$

$$= -(\pi \cos \pi - (-\pi)\cos(-\pi)) + \underbrace{\left[\sin x\right]_{-\pi}^{\pi}}_{=0}$$

$$= -2\pi \cos \pi = -2\pi(-1) = 2\pi$$

<u>Note</u>: This is b_1 in the Fourier expansion for $f(x) = x \quad (-\pi < x < \pi)$

2. $f(x) = x \qquad \|f\|_2 = (f \odot f)^{1/2}$

$$f \odot f = \int_{-\pi}^{\pi} x x \, dx = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = 2\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3} \cdot \pi^3$$

$$\Rightarrow \|f\|_2 = \sqrt{\frac{2}{3}\pi^3} = \sqrt{\frac{2}{3}}\, \pi^{3/2}$$

$$f(x) = e^x \quad f \odot f = \int_{-\pi}^{\pi} e^x e^x dx = \int_{-\pi}^{\pi} e^{2x} dx = \left[\frac{e^{2x}}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[e^{2\pi} - e^{-2\pi}\right]$$

$$\|f\|_2 = \frac{1}{\sqrt{2}}\left(e^{2\pi} - e^{-2\pi}\right)^{1/2}.$$

$$e^{inx} \odot e^{imx} = \int_{-\pi}^{\pi} e^{inx} \overline{e^{imx}} \, dx = \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx$$

$$= \int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

3a) If $n \neq m$, $e^{inx} \odot e^{imx} = \left[\frac{e^{i(n-m)x}}{i(n-m)}\right]_{-\pi}^{\pi} = \frac{e^{i(n-m)\pi} - e^{i(n-m)(-\pi)}}{i(n-m)} = 0$

because $(n-m)\pi$ and $(n-m)(-\pi)$ differ by an integer multiple of 2π. [namely, $2(n-m)\pi$]

If $n = m$,

$$\left(\|e^{inx}\|_2\right)^2 = e^{inx} \odot e^{inx} = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

hence

$$\|e^{inx}\|_2 = \sqrt{2\pi}$$

$$= \frac{2}{2\pi i n}\left(\left[\frac{x e^{-inx}}{(-in)}\right]_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi} \frac{e^{-inx}}{(-in)}\,dx}\right)$$

$$= \frac{1}{(-in)}\,(1 \odot e^{inx}) = 0$$

since $1 = e^{i0x} \perp e^{inx}$

$$= \frac{1}{\pi n^2}\cdot\left(\pi e^{-in\pi} - (-\pi)e^{in\pi}\right) \qquad e^{in\pi} = e^{-in\pi} = (-1)^n$$

$$= \frac{1}{n^2}\, 2\cdot(-1)^n$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2}{2\pi}\frac{\pi^3}{3} = \frac{\pi^2}{3}$$

$$x^2 = \frac{\pi^2}{3} + 2\sum_{n\neq 0} \frac{(-1)^n e^{inx}}{n^2}$$

2. Simplification:

Let $x = \sum c_n e^{inx}$ (in book)

$$x^3 = \sum_{-\infty}^{\infty} d_n e^{inx} \qquad \text{(you have to find } d_n\text{)}$$

Then $\frac{1}{3}(x - \pi^2 x^3) = \sum \left(\frac{1}{3} c_n - \pi^2 d_n\right) e^{inx}$.