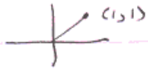
Assignment 1

1. a) $\dfrac{(1+i)(3-2i)}{8+i} = \dfrac{(3-(-2))+(-2+3)i}{8+i} = \dfrac{(5+i)(8-i)}{(8+i)(8-i)} = \dfrac{(40+1)+(-5+8)i}{8^2+1^2}$

$= \dfrac{41+3i}{9} = 41 + \frac{1}{3}i$

b) $1+i$



$r = \sqrt{2},\ \theta = \frac{\pi}{4}$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

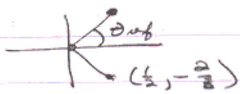
$7+2i$

$r = \sqrt{49+4} = \sqrt{53}$

$\theta = \tan^{-1}\frac{2}{7} \cong .2783$ (radians)

$7+2i = \sqrt{53}\,(\cos .2783 + i\sin .2783)$

$\frac{1}{2} - \frac{2}{3}i$



$r = \sqrt{\frac{1}{4} + \frac{4}{9}} = \sqrt{\frac{9+16}{36}} = \frac{5}{6}$

$\tan\theta_{ref} = \frac{y}{x} = \dfrac{\frac{2}{3}}{\frac{1}{2}} = \frac{4}{3}$

$\theta_{ref} = .9273 \qquad \theta = -.9273$ or $-.9273 + 2\pi = 5.356$

$\frac{1}{2} - 3i = \frac{5}{6}\left[\cos(-.9273) + i\sin(-.9273)\right]$

c) $z^2 = 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z = r\cos\theta$

$z^2 = r^2\cos 2\theta = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$r^2 = \sqrt{2} \Rightarrow r = \sqrt[4]{2}$

$2\theta = \frac{\pi}{4} \Rightarrow \theta_1 = \frac{\pi}{8} \qquad 2\theta = \frac{\pi}{4} + 2\pi \Rightarrow \theta_2 = \frac{\pi}{8} + \pi = \frac{9\pi}{8}$

$z_1 = \sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

$z_2 = \sqrt[4]{2}\left(\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)\right)$

d) $\overline{zw} = \overline{(x+iy)(u+iv)} = \overline{(xu-yv)+i(xv+yu)}$

$= (xu-yv) - i(xv+yu)$

$\bar{z}\,\bar{w} = (x-iy)(u-iv) = xu - yv + i(-xv-yu)$

$|zw|^2 = (zw)\overline{(zw)} = (z\bar{z})(w\bar{w}) = |z|^2|w|^2$

$\Rightarrow |zw| = |z||w|.$

2. p195 : 1 $\lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\right|\left|\frac{n!}{z^n}\right| = \lim_{n\to\infty}\frac{|z|}{n+1} = 0 < 1$

for all z so series converges for all z.

2. $\cos z = \frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\left[1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots\right.$

$\left. + \; 1 - iz + \frac{(-iz)^2}{2!} + \frac{(-iz)^3}{3!} + \frac{(-iz)^4}{4!} + \cdots\right]$

$= \frac{1}{2}\left[1 + \cancel{iz} - \frac{z^2}{2} - \cancel{\frac{iz^3}{3!}} + \frac{z^4}{4!} + \cdots\right.$

$\left. + \; 1 - \cancel{iz} - \frac{z^2}{2} + \cancel{\frac{iz^3}{3!}} + \frac{z^4}{4!} + \cdots\right]$

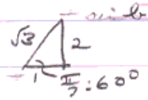
$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

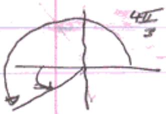
13 a) $e^{1+i} = e^1 e^{i1} = e(\cos 1 + i \sin 1) = (e\cos 1) + i\, e \sin 1$

b) $\sin(a+bi) = \sin a \cos ib + \cos a \sin ib$ (Theorem 50a)

$= \sin a \cosh b + i \cos a \sinh b$ "

$\sin\left(\frac{4\pi}{3} + i\right) = \sin\frac{4\pi}{3}\cosh 1 + i\cos\frac{4\pi}{3}\sinh 1$



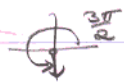


$\sin\frac{4\pi}{3} = -\sin\left(\frac{4\pi}{3} - \pi\right) = -\sin\frac{\pi}{3} = -\frac{2}{\sqrt{3}}$

$\cos\frac{4\pi}{3} = -\cos\left(\frac{4\pi}{3} - \pi\right) = -\cos\frac{\pi}{3} = \frac{-1}{\sqrt{3}}$

$\cosh 1 = \frac{e^1 + e^{-1}}{2}$ $\sinh 1 = \frac{e^1 - e^{-1}}{2}$

$= -\frac{2}{\sqrt{3}}\left(\frac{e + e^{-1}}{2}\right) + i\frac{1}{\sqrt{3}}\left(\frac{e^1 - e^{-1}}{2}\right)$



15. $\log z = \log(re^{i\theta}) = \log r + i\theta$

$-4 = 4(\cos\pi + i\sin\pi) = 4e^{i\pi}$ | $-i = e^{i3\pi/2}$

$\log(-4) = \log 4 + i\pi + 2n\pi$ | $\log(-i) = \frac{3\pi}{2}i$

$\boxed{\text{Lets assume } m, n > 0}$ (note $\cos(-m)x = \cos mx$, $\sin(-n)x = -\sin nx$

Case $\boxed{\text{Use } \int_{-\pi}^{\pi} e^{i\beta x}\,dx = 0 \text{ for } \beta \neq 0}$

$m \neq n$ $\quad \int_{-\pi}^{\pi} \cos mx \sin nx \, dx = \int_{-\pi}^{\pi} \left(\frac{e^{imx} + e^{-imx}}{2}\right)\left(\frac{e^{inx} - e^{-inx}}{2i}\right) dx$

$$= \frac{1}{4i}\Big[\underbrace{\int_{-\pi}^{\pi} e^{i(m+n)x}\,dx}_{=0} + \underbrace{\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx}_{=0} + \underbrace{\int_{-\pi}^{\pi} e^{i(-m+n)x}\,dx}_{=0} - \underbrace{\int_{-\pi}^{\pi} e^{-i(m+n)x}\,dx}_{=0}$$

$= 0$

since $m+n, m-n, -m+n, -(m+n) \neq 0$.

$m = n$ $\quad \int_{-\pi}^{\pi} \cos mx \sin mx \, dx =$ (as above)

$$= \frac{1}{4i}\Big[\underbrace{\int_{-\pi}^{\pi} e^{i2mx}\,dx}_{=0} - \underbrace{\int_{-\pi}^{\pi} e^{0x}\,dx + \int_{-\pi}^{\pi} e^{0}\,dx}_{=0} - \underbrace{\int_{-\pi}^{\pi} e^{-i2mx}\,dx}_{=0}$$

$= 0.$