Assignment 6

1. $f(x)=x^2-x^3 \quad 0\le x\le 1$ $\qquad$ <u>max & min</u> occur at endpoint or at a critical point.

$0=f'(x)=2x-3x^2=x(2-3x)$

$\Rightarrow x=0,\ 2-3x=0 \Rightarrow x=2/3$

<u>Check</u> f at $0, 1, 2/3$

x	$f(x)$
0	0
1	0
2/3	$\frac{4}{9}-\frac{8}{27}=\frac{12-8}{27}=\frac{4}{27}$

$\|f\|_\infty=\frac{4}{27}$

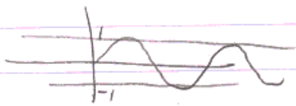
$f(x)=x^2-x^3 \quad -1\le x\le 2$

$0=f'(x)\Rightarrow x=0, 2/3$

Check f at $-1, 0, 2/3, 2$

x	$f(x)$
-1	$1+1=2$
0	0
2/3	4/27
2	$4-8=-4$

$\|f\|_\infty=\sup_{-1\le x\le 2}|f(x)|=|-4|=4.$



$f(x)=\sin x \quad -\infty<x<\infty$

$\|f\|_\infty=1$

2. p. 163.1 $f_n(x)=\frac{x}{1+nx}$ $\qquad I=[0,1],\ f=0$

$f_n(x)-f(x)=f_n(x)$

$f_n'(x)=\frac{1(1+nx)-xn}{(1+nx)^2}=\frac{1}{(1+nx)^2}$ NOWHERE ZERO (no critical pt!)

check f_n at endpts $f_n(0)=0,\ f_n(1)=\frac{1}{1+n}$

$\|f_n-f\|_\infty=\|f_n\|_\infty=\frac{1}{1+n}$

$\boxed{-\frac{1}{1+n}<-\frac{1}{2n}<0}$

$\Rightarrow \lim_{n\to\infty}\|f_n-f\|=0$, so unif convergent

3. $f_n(x)=\frac{nx^2}{1+nx}$ $\qquad f(x)=x$

x	$g_n(x)$
0	0
$\frac{1}{n}$	$-\frac{1}{2n}$
1	$-\frac{1}{1+n}$

$g_n(x)=(f_n-f)(x)=\frac{nx^2-x(1+nx)}{1+nx}=\frac{-x}{1+nx}$

$0=g_n'(x)=\frac{(-1)-(-x)n}{(1+nx)^2}=\frac{nx-1}{(1+nx)^2}\Rightarrow x=\frac{1}{n}$

$g_n(\frac{1}{n})=\frac{\frac{1}{n}-\frac{1}{n}(1+1)}{2}=-\frac{1}{2n}$ $\qquad \|g_n\|_\infty=\frac{1}{1+n}\to 0.$

12 $f_n(x) = n^2 x^n (1-x) = n^2 x^n - n^2 x^{n+1}$

$f(x) = 0 \qquad I = [0,1]$

$$0 = f_n'(x) = n^2 \cdot n x^{n-1} - n^2 (n+1) x^n$$

$$= n^3 x^{n-1} - n^3 x^n - n^2 x^n$$

$$= x^{n-1}(n^3 - n^3 x - n^2 x)$$

$x = 0, \quad (n^3 + n^2) x = n^3 \Rightarrow x = \frac{n^3}{n^3 + n^2} = \frac{n}{n+1}$

$f_n(0) = 0, \; f_n(1) = 0 \qquad f_n\left(\frac{n}{n+1}\right)$

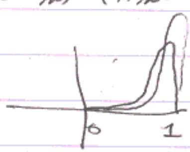
$$\|f_n\|_\infty = \max\{\underbrace{|f_n(0)|, |f_n(1)|}_{0}, |f_n(\tfrac{n}{n+1})|\}$$

$$= |f_n(\tfrac{n}{n+1})| = \frac{n^2 n^n}{(n+1)^n}\left(1 - \frac{n}{n+1}\right)$$

$$= \frac{n^2 n^n}{(n+1)^n} \cdot \frac{1}{(n+1)} = \frac{n^{n+2}}{(n+1)^{n+1}}$$

$$= \frac{n}{(1+\frac{1}{n})^{n+1}} = \frac{n}{(1+\frac{1}{n})^n \cdot (1+\frac{1}{n})} \to \infty \quad \text{since} \left\{ \begin{array}{l} 1+\frac{1}{n} \to 1 \\ (1+\frac{1}{n})^n \to e \end{array} \right.$$

Does not converge uniformly.



16. $f_n(x) = (n+1)(n+2) x^n (1-x)$

$\underline{x<1}$ $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (n+1)(n+2) x^n (1-x) = (1-x) \lim (n+1)(n+2) x^n$

Say $x = \frac{1}{c}$, $c > 1$. $\lim_{n\to\infty} \frac{(n+1)(n+2)}{c^n}$

$$= \lim_{x\to\infty} \frac{(x+1)(x+2)}{c^x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{2x+3}{c^x \ln c} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{2}{c^x (\ln c)^2} = \frac{2}{\infty} = 0$$

Thus $f_n(x) \to 0$ for each x in I! On the other hand

$$\int_0^1 f_n(x)\,dx = (n+1)(n+2)\int_0^1 (x^n - x^{n+1})\,dx$$

$$= (n+1)(n+2)\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$= (n+1)(n+2)\,\frac{(n+2)-(n+1)}{(n+1)(n+2)} = 1$$

whereas $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$. The convergence <u>cannot</u> <u>be uniform</u> (since $f_n \to f$ <u>unif</u> $\Rightarrow \int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$)

i.e., without doing any calculation, we know that $\|f_n - f\|_\infty$ does not converge to 0.

7.167:1 We have $h < 1$ and $I = [-h, h] \Rightarrow$

$$\|x^n\|_\infty = \sup_{-h \le x \le h} |x^n| \le h^n$$

$$\sum \|x^n\|_\infty = \sum h^n < \infty \quad \text{(geometric series ratio } h<1\text{)}$$

<u>You can use following result for power series</u>

If $\sum c_n x^n$ converges on $(-r, r)$ then it converges uniformly on $I = [-h, h]$ for any $h < r$.

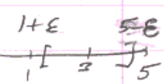
<u>Proof:</u> $\|c_n x^n\|_\infty = \sup_{-h \le x \le h} |c_n x^n| = |c_n|\,|h|^n$

$$\sum \|c_n x^n\|_\infty = \sum |c_n|\,|h|^n < \infty$$

— we are using the fact that if a power series converges on $(-r, r)$ then it converges <u>absolutely</u> at each point in $(-r, r)$.

7. $\sum \frac{n!(x-3)^n}{1\cdot3\cdot5\cdots(2n-1)}$ ratio test will give you radius of convergence

$$\lim \left| \frac{(n+1)!(x-3)^{n+1}}{1\cdot3\cdot5\cdots(2n-1)(2n+1)} \cdot \frac{1\cdot3\cdot5\cdots(2n-1)}{n!(x-3)^n} \right|$$



$$= |x-3| \lim \frac{n+1}{2n+1} = |x-3|\cdot\frac{1}{2} < 1$$

$|x-3| < 2$ $\quad I = [1+\varepsilon, 5-\varepsilon]$