Assignment 4

1. p. 129:4 NOTE "$\log x = \ln x$" (this is not $\log_{10} x$) $\boxed{a_n = \frac{f^{(n)}(1)}{n!}}$

$f(x) = \log(1+x)$ $\qquad f(1) = \log 2$ $\qquad a_0 = \log 2$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\qquad f'(1) = \frac{1}{2}$ $\qquad a_1 = \frac{1}{2}\cdot\frac{1}{1!}$

$f''(x) = (-1)(1+x)^{-2}$ $\qquad f''(1) = -\frac{1}{2^2}$ $\qquad a_2 = -\frac{1}{2^2}\,\frac{1}{2!}$

$f'''(x) = (-2)(-1)(1+x)^{-3}$ $\qquad f'''(1) = 2!\,\frac{1}{2^3}$ $\qquad a_3 = \frac{2!}{3!}\cdot\frac{1}{2^3}$

$f^{(4)}(x) = (-3)(-2)(-1)(1+x)^{-4}$ $\qquad f^{(4)}(1) = -3!\,\frac{1}{2^4}$ $\qquad = \frac{1}{3}\cdot\frac{1}{2^3}$

$= (-1)^3\, 3!\,(1+x)^{-4}$ $\qquad$ etc.

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$ $\qquad f^{(n)}(1) = (-1)^{n-1}(n-1)!\,\frac{1}{2^n}$ $\qquad a_n = (-1)^{n}\,\frac{1}{n\,2^n}$

$$f(x) = \sum a_n (x-a)^n = \log 2 + \frac{1}{2}(x-1) + \frac{1}{2^2\,2}(x-1)^2 + \frac{1}{2^3\,3}(x-1)^3$$

$$- \dots + \frac{(-1)^n}{n\,2^n}(x-1)^n + \dots$$

$$= \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}(x-1)^n$$

8. $f(x) = e^x$ $\qquad f(1) = e$ $\qquad a_0 = e$

$f'(x) = e^x$ $\qquad f'(1) = e$ $\qquad a_1 = \frac{e}{1!}$

$f''(x) = e^x$ $\qquad f''(1) = e$ $\qquad a_2 = \frac{e}{2!}$

$\vdots$

$f^{(n)}(x) = e^x$ $\qquad f^{(n)}(1) = e$ $\qquad a_n = \frac{e}{n!}$

$$e^x = f(x) = e + e(x-1) + e\frac{(x-1)^2}{2!} + \frac{e(x-1)^3}{3!} + \dots$$

<u>CHECK</u>: $e^x = e\cdot e^{x-1} = e\left[1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \dots\right]$

16. $f(x) = x^m$ $\qquad f(1) = 1$ $\qquad a_0 = 1$

$f'(x) = m x^{m-1}$ $\qquad f'(1) = m$ $\qquad a_1 = \frac{m}{1!}$

$f''(x) = m(m-1)\,x^{m-2}$ $\qquad f''(1) = m(m-1)$ $\qquad a_2 = \frac{m(m-1)}{2!}$

$\vdots$

$f^{(m)}(x) = m!$ $\qquad f^{(m)}(1) = m!$ $\qquad a_m = \frac{m(m-1)\dots 1}{m!}$

$f^{(m+1)} = f^{(m+2)} = \dots = 0$

$$x^m = 1 + m(x-1) + \frac{m(m-1)}{2!}(x-1)^2 + \dots + \frac{m!}{m!}(x-1)^m$$

(CHECK): $x^m = [1+(x-1)]^m$ USE BINOMIAL THM.

37.2 $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n(x)$

$$|R_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}\, x^{n+1}\right| \qquad -.4 < \xi < 0 \qquad e^\xi < e^0 = 1$$

$$R_n(-.4) = \left|\frac{e^\xi(-.4)^{n+1}}{(n+1)!}\right| < \frac{1\cdot(.4)^{n+1}}{(n+1)!}$$

We want to find n such $|R_n(-.4)| < 10^{-4}$. "Trial & Error"

$n=1$ $|R_1(-.4)| < \frac{(.4)^2}{2!} = \frac{.16}{2} = .08$

$$|R_3(-.4)| < \frac{(.4)^4}{4!} < 1.07\times10^{-3}$$

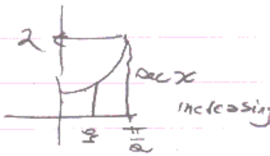
$$R_4(-.4) < \frac{(.4)^5}{5!} < 8.6\times10^{-5}$$

So the approx to $e^{-.4}$ is

$$1 + (-.4) + \frac{(-.4)^2}{2!} + \frac{(-.4)^3}{3!} + \frac{(-.4)^4}{4!} = .6704$$

"EXACT" VALUE: $e^{-.4} = .6720320046$

6. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$ (see p. 129)

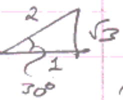


$$|R_1(x)| = \left|\frac{f''(\xi)\,x^2}{2!}\right| \qquad x = .1 \qquad 0 < \xi < .1$$
$$\Rightarrow 0 < \xi < \frac{\pi}{6}$$

$$= \frac{2\sec^2\xi\,\tan\xi\,(.1)^2}{2} < \frac{\sec^2\frac{\pi}{6}\,\tan\frac{\pi}{6}\,(.01)}{2}$$

$\sec\frac{\pi}{6} = \left(\cos\frac{\pi}{6}\right)^{-1} = 2$ $\qquad < \frac{2}{2}\frac{\sqrt{3}}{2}(.01) < \frac{1.8}{2}\times.01$

$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{2}$



Desired approx to $\tan(.1)$ is $\quad \tan x \approx x = .1$

"Exact" value $\tan(.1) = .10033\dots$

NOTE We could have also used $R_2(.1)$ since $\tan x = x + 0x^2 + \frac{x^3}{3} + \dots$

3. p. 144 : 1

$$f(x) = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

4. $$f(x) = \frac{1}{(1+x)^2} = (1+x)^{-2} = -\frac{d}{dx}(1+x)^{-1}$$

$$= \frac{d}{dx}[1 - x + x^2 - x^3 + x^4 - \dots]$$

$$= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^n}{(-1)^n n x^{n-1}}\right| = \lim_{n\to\infty}(1+\tfrac{1}{n})|x| = |x|$$

converges if $|x|<1$, diverges if $|x|>1$

13. $$\sum_{k=1}^{\infty} k x^{k-1} = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right) = \frac{d}{dx}\frac{1}{1-x} = +\frac{1}{(1-x)^2}$$

14. Let $g(x) \overset{(A)}{=} x\sum_{k=1}^{\infty}\frac{x^k}{k(k+1)} = \sum_{k=1}^{\infty}\frac{x^{k+1}}{k(k+1)}$

Then $g'(x) \overset{(B)}{=} \sum_{k=1}^{\infty}\frac{x^k}{k}$ $\quad g''(x) = \sum_{k=1}^{\infty} x^{k-1} = 1 + x + \dots = \frac{1}{1-x}$

Thus $g'(x) = \int\frac{dx}{1-x}$ Let $u = \ln(1-x) \Rightarrow du = \frac{-1}{1-x}$

$$\Rightarrow g'(x) = -\int du = -u + C = -\ln(1-x) + C$$

But from (B), $g'(0) = 0$, so

$$-\ln(1-0) + C = 0 \Rightarrow C = \ln 1 = 0, \text{ and}$$

$g'(x) = -\ln(1-x)$. In turn

$$g(x) = -\int\ln(1-x)\,dx \quad \text{Let } v = 1-x \quad dv = -dx$$

$$= \int \ln v\,dv = v\ln v - v + D \text{ (integration by parts)}$$

$$= (1-x)\ln(1-x) - (1-x) + D$$

From (A), $g(0)=0$, so

$$0 = g(0) = (1-0)\ln(1-0) - (1-0) + D = -1 + D \Rightarrow D =$$

$$g(x) = (1-x)\ln(1-x) + x$$

From (A), $$\sum_{0}^{\infty}\frac{x^k}{k(k+1)} = \frac{1}{x}g(x) = \frac{1}{x}[(1-x)\ln(1-x) + x]$$