Assignment 3

2. $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ diverges (p-test; $p = \frac{1}{2} \le 1$)

4. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{n}{n\sqrt{n}} = \sum \frac{1}{\sqrt{n}} = \infty$ (see 2) $\underline{\text{diverges}}$ comparison test

6. $\sum_{n=1}^{\infty} \frac{1}{n\, 2^n} \le \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$ converges (comparison test)

$\underline{\underline{\text{OR}}}$ $\rho = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{n\, 2^n}{(n+1) 2^{n+1}} = \lim_{n \to \infty} \frac{1}{(1+\frac{1}{n}) \cdot 2} = \frac{1}{2} < 1$

converges by ratio test

12. $\sum_{n=1}^{\infty} \frac{2}{2^n + 3} \le \sum_{n=1}^{\infty} \frac{2}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \underbrace{1 + \frac{1}{2} + \frac{1}{4} + \cdots}_{\text{geometric series}} = 2$

converges by comparison test

31. $\int_2^{\infty} \frac{dx}{x(\log x)^p} = \lim_{R \to \infty} \int_2^R \frac{dx}{x(\log x)^p} = \lim_{R \to \infty} \left[\int \frac{dx}{x(\log x)^p} \right]_2^R$

Let $u = \log x$, $du = \frac{dx}{x}$ $\int \frac{dx}{x(\log x)^p} = \int \frac{du}{u^p}$

If $p \neq 1$, $\int \frac{dx}{x(\log x)^p} = \int \frac{du}{u^p} = \frac{u^{-p+1}}{-p+1} = \frac{(\log x)^{-p+1}}{-p+1}$

$\int_2^R \frac{dx}{x(\log x)^p} = \frac{(\log R)^{-p+1}}{-p+1} - \frac{(\log 2)^{-p+1}}{-p+1}$

If $p > 1$, $\lim_{R \to \infty} \int_2^R \frac{dx}{x(\log x)^p} = \lim_{R \to \infty} \left[\frac{1}{(\log 2)^{p-1}} - \frac{1}{(\log R)^{p-1}} \right]$

$= \frac{1}{(\log 2)^{p-1}} - \frac{1}{\infty} = \frac{1}{(\log 2)^{p-1}}$

since $p > 1$

If $p < 1$

$\lim \int_2^R \frac{dx}{x(\log x)^p} = \lim (\log R)^{1-p} - (\log 2)^{1-p}$

since $p < 1$

$= \infty - (\log 2)^{1-p} = \infty$

If $p = 1$, $\int \frac{dx}{x \log x} = \int \frac{du}{u} = \log \log x$

$\int_2^{\infty} \frac{dx}{x \log x} = \lim_{R \to \infty} \log \log R - \log \log 2 = \infty$

Note that $f(x) = \frac{1}{x(\log x)^p}$ is decreasing since

$$f'(x) = -\frac{1}{(x\log x)^2} \cdot (x\log x)' = -\frac{\log x + \frac{x}{x}}{(x\log x)^2} < 0$$

for $x > 1$

Hence we may use integral test:

a) $p \le 1$ divergent b) $p > 1$ convergent

p.119: 2. $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \lim_{n\to\infty} \frac{10}{n+1} = 0 <$

convergent

8. $\sum\left|\frac{(-1)^n}{n^p}\right| = \sum \frac{1}{n^p} = \infty$ for $p < 1$ not absolutely convergent

Since $\frac{1}{1^p} \ge \frac{1}{2^p} \ge \frac{1}{3^p} \ldots$ and $\lim \frac{1}{n^p} = 0$,

the series converges by the alternating series test. Thus we have the series "converges conditionally"

26. $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{(n+1)\log(n+1)}\right| = \sum_{n=2}^{\infty} \frac{1}{n\log n}$ (write out a few terms)

and we saw above this diverges. Since $\frac{1}{(n+1)\log(n+1)}$ decreases and has limit 0, it converges by the alternating series test (thus cond. convergence)

32. Since $\sum_{n=1}^{\infty} u_n$ converges, $\lim_{n\to\infty} u_n = 0$. It follows that

we may find an N such that $n \ge N$ implies $|u_n - 0| < 1$ (let $\varepsilon =$ in the definition of "limit"), i.e., $u_n < 1$. In general $0 \le x < 1$ $\Rightarrow x^{p-1} < 1$ (since $p > 1$) $\Rightarrow x^p = x \cdot x^{p-1} < x$. Thus

$$n \ge N \Rightarrow u_n^p < u_n$$

and $\sum_N^\infty u_n^p < \sum_N^\infty u_n < \infty$. Thus

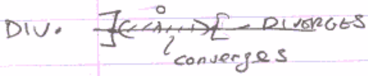
$$\sum_1^\infty u_n^p = \underbrace{\left(\sum_1^{N-1} u_n^p\right)}_{\text{finite}} + \sum_N^\infty u_n^p < \infty.$$

p. 125 2. $\rho = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{(-1)^n x^n}\right| = |x|$

Converges if $\rho = |x| < 1$ Diverges if $\rho = |x| > 1$ (RATIO TEST)

$x=1$ $\sum (-1)^n 1^n$ diverges $(-1)^n \not\to 0$

$x=-1$ $\sum (-1)^n(-1)^n$ " $1^n = 1 \not\to 0$



12. $\rho = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,(x+1)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n!\,(x+1)^n}\right|$

$$= \lim_{n\to\infty} \frac{n+1}{5}|x+1| = \begin{cases} \infty & \text{if } x \neq -1 \\ 0 & \text{if } x = -1 \end{cases}$$

converges if $x = -1$ $(\rho = 0)$

diverges if $x \neq -1$ $(\rho = \infty)$

No endpoints to check.

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converges

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No endpoints to check.