Math 245b (W03)

In case you missed some of the details, here is the crucial part of our proof that monotone functinos are differentiable almost everywhere (with respect to Lebesgue measure).

Theorem: Suppose that $F$ is a monotone increasing) function on $[a, b]$. Then $F'(x)$ exists for almost all $x \in [a, b]$, it is integrable, and $\int_a^b F'(x)dx \leq F(b) - F(a)$.

Proof: We begin by showing that $\overline{D}^- F(x) = \underline{D}^- F(x) = \overline{D}^+ F(x) = \underline{D}^+ F(x)$ for almost all $x$. Suppose for example that $\lambda^*(\left\{x : \overline{D}^- F(x) < \overline{D}^+ F(x)\right\}) \neq 0$ (you should consider some of the other possibilities). Since

$$\left\{x : \overline{D}^- F(x) < \overline{D}^+ F(x)\right\} = \bigcup_{r<s, r,s\in\mathbb{Q}} E_{r,s}$$

where

$$E_{r,s} = \left\{x : \overline{D}^- F(x) < r < s < \overline{D}^+ F(x)\right\},$$

we must have that $\lambda^*(E_{r,s}) > 0$ for some $r < s$. Given $\varepsilon > 0$, let $G$ be open with $E_{r,s} \subseteq G$ and $\lambda(G) < \lambda(E_{r,s}) + \varepsilon$. If $x \in E_{r,s}$, we have that there exist arbitrarily small $h > 0$ such that

$$\frac{F(x) - F(x-h)}{h} < r.$$

Let $\mathcal{I}$ be the set of all intervals $I = [x-h, x] \subseteq G$ such that $F(x) - F(x-h) < rh$. This is a Vitalli cover of $E_{r,s}$, hence we may find $I_i = [x_i - h_i, x_i] \in \mathcal{I}$ with

$$\lambda^*(E \backslash [I_1 \sqcup \ldots \sqcup I_m]) < \varepsilon,$$

and thus

$$\lambda^*(E \cap [I_1 \sqcup \ldots \sqcup I_m]) \geq \lambda^*(E) - \varepsilon$$

(remember that since $I_1 \sqcup \ldots \sqcup I_m$ is measurable, it "splits" arbitary sets). Summarizing, the "slow" increase of $F$ caused by $\overline{D}^- F(x) < r$ enable us to conclude that

$$\begin{aligned} \sum F(x_i) - F(x_i - h_i) \leq \sum r h_i &= r\lambda(I_1 \sqcup \ldots \sqcup I_m) \\ &\leq r\lambda(G) < r(\lambda^*(E_{r,s}) - \varepsilon). \end{aligned}$$

Next we will take advantage of the "fast" increase of $F$ caused by $s < \overline{D}^+ F(x)$ to lead us to a contradiction. Let $I_i^o = (x_i - h_i, x_i)$ (we want some "wiggle" room). Then if we let $D = E \cap [I_1^o \sqcup \ldots \sqcup I_m^o]$, we have that $\lambda^*(D) \geq \lambda^*(E) - \varepsilon$ (you check that eliminating finitely many points from a set doesn't change its outer measure: remember the "splitting" principle). For each $y \in D$ we may find arbitrarily small $k > 0$ such that

$$\frac{F(y+k) - F(y)}{k} > s.$$

It follows that the collection $\mathcal{J}$ of all intervals $J = [y, y+k]$ with $y \in D$ and $J \subseteq I_1^o \sqcup \ldots \sqcup I_m^o$ for which $F(y+k) - F(y) > sk$ is a Vitalli cover of $D$, and we may select $J_j = [y_j, y_j + k_j] \in \mathcal{J}$ with

$$\lambda^*(D \cap [J_1 \sqcup \ldots \sqcup J_n]) \geq \lambda^*(D) - \varepsilon \geq \lambda^*(E_{r,s}) - 2\varepsilon.$$

We conclude that

$$\sum F(y_j + k_j) - F(y_j) > \sum sk_j = s\lambda(J_1 \sqcup \ldots \sqcup J_n) \geq s(\lambda^*(E_{r,s}) - 2\varepsilon).$$

Since each $J_j$ lies in a unique $I_i$, and $F$ is increasing,

$$\begin{aligned} \sum F(y_j + k_j) - F(y_j) &= \sum_i \sum_{\{j: I_j \subseteq I_i\}} F(y_j + k_j) - F(y_j) \\ &\leq \sum_i F(x_i) - F(x_i - h_i). \end{aligned}$$

To understand the last inequality, note that if, for example,

$$[y_1, y_1 + k_1] \sqcup [y_2, y_2 + k_2] \subseteq [x_1 - h, x_1]$$

then from the monotonicity of $F$ and the inequalities

$$x_1 - h \leq y_1 < y_1 + k_1 < y_2 < y_2 + k_2 < x_1$$

$F(x_1) - F(x_1 - h)$ can be written as the telescoping sum

$$(F(y_1) - F(x_1 - h)) + (F(y_1 + k_1) - F(y_1)) + \cdots$$

which is greater than or equal to $(F(y_1 + k_1) - F(y_1)) + (F(y_{21} + k_2) - F(y_2))$.

Putting our inequalities together,

$$s(\lambda^*(E_{r,s}) - 2\varepsilon) \leq r(\lambda^*(E_{r,s}) - \varepsilon).$$

Since we may make $\varepsilon$ arbitrarily small, we conclude that

$$s\lambda^*(E_{r,s}) \leq r\lambda^*(E_{r,s})$$

which is impossible since $s > r$.

From the previous argument (strictly speaking one has to prove all four expressions are equal almost every where, and thus one has to prove eight strict inequalities can't happen - some of these are automatic since lim inf $\leq$ lim sup) the set $E = \{x : F'(x) \text{ does not exist}\}$ satisfies $\lambda^*([a, b] \backslash E) = 0$, and thus it is measurable, as is its complement $D = \{x : F'(x) \text{ exists}\}$. We extend $F$ to a monotone function on $[a, b + 1]$ by letting $F(x) = F(b)$ for $b \leq x \leq b + 1$. For each $n \in \mathbb{N}$, let

$$g_n(x) = \frac{F(x + 1/n) - F(x)}{1/n}.$$

If $x \in D$, we have that

$$\lim_{n \to \infty} g_n(x) = F'(x).$$

Since $F$ is monotonic, it is Borel (see class notes) and thus $g_n$ is Borel. It follows that the set $D_0 = \{x \in [a, b] : g_n(x) \text{ converges}\}$ is Borel and the function

$$g(x) = \left\{ \begin{array}{cc} \lim_{n \to \infty} g_n(x) & \text{if } x \in D_0 \\ 0 & \text{if } x \notin D_0 \end{array} \right\}$$

is Borel and equals $F'(x)$ for almost all $x$. If we interpret $F'(x)$ to be 0 for $x \notin D$, we see that $F'(x) = g(x)$ for all $x$, and thus $F'(x)$ is measurable (see homework problem).

Finally since $g_n \geq 0$ ($F$ is monotone increasing), we may use Fatou's lemma:

$$\int_a^b F'(x)dx = \int_a^b g(x)dx \leq \underline{\lim} \int_a^b g_n(x)dx.$$

But we have that

$$\begin{aligned}
\int_a^b \frac{F(x+1/n)-F(x)}{1/n}dx &= n\int_a^b F(x+1/n)-F(x)dx \\
&= n\int_{a+1/n}^{b+1/n} F(x)dx - n\int_a^b F(x)dx \\
&= n\int_b^{b+1/n} F(x)dx - n\int_a^{a+1/n} F(x)dx \\
&\le n\int_b^{b+1/n} F(b)dx - n\int_a^{a+1/n} F(a)dx \\
&= F(b)-F(a)
\end{aligned}$$

where we used $F(x) = F(b)$ for $x \geq b$ and $F(x) \geq F(a)$ for all $x \in [a, a+1/n]$. We conclude that

$$\int_a^b F'(x)dx \leq F(b) - F(a).$$

In particular since the right side is finite, we conclude that $F'(x)$ is integrable, and in particular, $0 \leq F'(x) < \infty$ for almost all $x$. QED