Handout 9: Riemann Integration (from lecture on 11/27)

In previous lecture we proved:

$(X, \mathcal{M}, \mu)$ complete

$f, g : X \longrightarrow \mathbb{R}$ $\mu(\{x : f(x) \neq g(x)\}) = 0$, f meas. $\Rightarrow g$ meas.

Recall: given $f \in \ell^\infty([a,b])$ and a partition
$P: \; a = t_0 < t_1 < \dots < t_n = b$
we define

$$L_P(f) = \sum m_i (t_i - t_{i-1}) \qquad m_i = \inf\{f(t) : t_{i-1} \le t \le t_i\}$$
$$U_P(f) = \sum M_i (t_i - t_{i-1}) \qquad M_i = \sup\{f(t) : t_{i-1} \le t \le t_i\}$$

Write $P \preceq Q$ if the points in P are among those in Q. We have

$$P \preceq Q \Rightarrow L_P(f) \le L_Q(f) \le U_Q(f) \le L_P(f). \qquad \text{(why?)}$$

We say f is <u>Riemann integrable</u> if $\forall \varepsilon > 0, \exists P$:
$$|U_P(f) - L_P(f)| < \varepsilon.$$

We then define $\int_R f = \sup\{L_P(f) : P \text{ a partition of } [a,b]\}$

To connect this with Lebesgue integral [with respect to the measure space $([a,b], \mathcal{M}_\lambda, \lambda)$],
let
$$\varphi_P = \sum m_i \mathbb{1}_{[t_{i-1}, t_i)},$$
$$\psi_P = \sum M_i \mathbb{1}_{[t_{i-1}, t_i)}.$$
It is clear that
$$\varphi_P \le f \le \psi_P$$
and $L_P(f) = \int \varphi_P \, d\lambda \qquad U_P(f) = \int \psi_P \, d\lambda.$

<u>Theorem</u>: If $f : [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable
Then it is Lebesgue integrable and $\int_R f = \int f$

<u>Proof</u>: Let $P_1, P_2, \dots$ be such that
$$U_{P_n}(f) - L_{P_n}(f) \longrightarrow 0, \qquad P_n : a = t_0^{(n)} < \dots < t_{k_n}^{(n)} = b.$$
We may assume $P_1 \preceq P_2 \preceq \dots$ so that $\varphi_{P_n} \uparrow$ and $\psi_{P_n} \downarrow$.
Let $\varphi_n = \varphi_{P_n}$, $\psi_n = \psi_{P_n}$ and let $\varphi_n \uparrow g$
$\psi_n \downarrow h$. We note that
$$m_0 \le \varphi_n \le \psi_n \le M_0 \qquad m_0 = \inf f(x) \quad M_0 = \sup f(x)$$
hence $m_0 \le g \le f \le h \le M_0$.

By monotone convergence

$$I(\varphi_n) \uparrow I(g)$$

[strictly speaking we should have $\varphi_n \geq 0$, but we can fix this by replacing φ_n by $C+\varphi_n$ for $C=-m_0$ if $m_0<0$] Similarly

$$I(\psi_n) \downarrow I(h)$$

[first consider $C-\psi_n \uparrow C-h$ where C is sufficiently large constant]. We have

$$I(h-g) = I(h) - I(g) = \lim I(\psi_n) - I(\varphi_n)$$
$$= \lim U_{P_n}(f) - L_{P_n}(f) = 0$$

and thus $h-g=0$ a.e. (almost everywhere i.e. $\lambda(\{x: h-g\neq 0\})=0$

Since $g \leq f \leq h$, $f=g$ a.e., and since $g=\lim \varphi_n$ is measurable, so is f. (see above).

$$L_{P_m}(f) \longrightarrow \int_{\mathcal{R}} f \quad \text{(see above)}$$
$$\parallel$$
$$\int \varphi_m \, d\lambda \uparrow, \int g = \int f$$

and thus $\int_{\mathcal{R}} f = \int f$.

Theorem: $f \in \ell^\infty([a,b])$ is Riemann integrable $\Leftrightarrow$ the set of discontinuities has λ-measure 0.

Proof: $\Rightarrow$) Let $\varphi_n \uparrow g$ and $\psi_n \downarrow h$ be defined as above. Let $N=\{x: g(x)\neq h(x)\}$ and let

$$P_n: a=t_0^{(n)}<\dots<t_{k_n}^{(n)}=b.$$

Let $T=\{t_i^{(n)}\}$, $M=N\cup T$. Since T is countable, $\lambda(M)=0$.

We claim f is continuous on $[a,b]\setminus M$.

Fix $x_0\in[a,b]\setminus M$. To show f is cont. at x_0 say that $\varepsilon>0$ is given. We may choose n such that $\psi_n(x_0)-\varphi_n(x_0)<\varepsilon$. Then consider partition P_n: we may assume that $x_0\in(t_i^{(n)}, t_i^{(n)})$. But $\varphi_n(x)=m_i^{(n)}$ and $\psi_n(x)=M_i^{(n)}$ for all x in this intervals and thus

$$m_i^{(n)} \le \begin{Bmatrix} f(x) \\ f(x_0) \end{Bmatrix} \le M_i^{(n)} \quad \forall x \in J = (t_{i-1}^{(n)}, t_i^{(n)})$$

in particular

where, $M_i^{(n)} - m_i^{(n)} = \psi_n(x_0) - \varphi_n(x_0) < \varepsilon$. It follows that

$$x \in J \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Or if we let $\delta = \min\{|x - t_{i-1}^{(n)}|, |x - t_i^{(n)}|\}$,

$$|x - x_0| < \delta \Rightarrow x \in J \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

$\Leftarrow$) We may assume that $P_1 \le P_2 \le \ldots$ also satisfies

$$\max |t_i^{(n)} - t_{i-1}^{(n)}| \to 0 \quad \text{as } n \to \infty$$

(why?) Define φ_n, ψ_n, g, h as before. We claim that if f is continuous at x_0 then

$$g(x_0) = f(x_0) = h(x_0).$$

Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

we may choose n such that $|t_i^{(n)} - t_{i-1}^{(n)}| < \delta$ for all i. Say that $x_0 \in [t_{i-1}^{(n)}, t_i^{(n)}]$. Then

$$x \in [t_{i-1}^{(n)}, t_i^{(n)}] \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$$\Rightarrow \begin{Bmatrix} |M_i^{(n)} - f(x_0)| \le \varepsilon \\ |f(x_0) - m_i^{(n)}| \le \varepsilon \end{Bmatrix} \Rightarrow |M_i^{(n)} - m_i^{(n)}| < 2\varepsilon$$

$\Rightarrow$ $|\psi_n(x_0) - \varphi_n(x_0)| \le 2\varepsilon$. It follows that for all $k \in \mathbb{N}$,

$$|\psi_{n+k}(x_0) - \varphi_{n+k}(x_0)| \le 2\varepsilon \quad (\text{recall } \psi_n \uparrow,\ \varphi_n \downarrow)$$

and thus $|h(x_0) - g(x_0)| \le 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary $h(x_0) = g(x_0)$.

It follows that

$$U_{P_n}(f) - L_{P_n}(f) = \int (\psi_n - \varphi_n)\, d\lambda \to \int (h - g)\, d\lambda = 0$$

(this uses monotone convergence theorem — see above)

and thus f is Riemann integrable.