Solutions to Hour exam.

1. See solution to practice exam.

2. We let $\mathcal{E}([a,b]) \subseteq \ell^\infty([a,b])$ be the linear space of step functions $f = \sum_{i=1}^n c_i \mathbf{1}_{[t_{i-1},t_i)^*}$, and we defined $I : \mathcal{E} = \mathcal{E}([a,b]) \to \mathbb{R}$ by $I(f) = \sum_{i=1}^n c_i(t_i - t_{i-1})$. We showed this is a bounded linear mapping, and thus it is uniformly continuous. It follows that I has a unique continuous extension $\bar{I} : \overline{\mathcal{E}} \to \mathbb{R}$. We then showed $C([a,b]) \subseteq \overline{\mathcal{E}}$ by using the fact that a continuous function on $[0,1]$ is uniformly continuous. Our integral was then just the restriction of $\bar{I}$ to $\overline{\mathcal{E}}$.

3 Let $g = \mathbf{1}_{\{0\}}$, and fix $\bar{t} \in (t_0, t_1)$. If $f \in \mathcal{E}$ and $\|g - f\|_\infty \leq \varepsilon$, then in particular, $|g(0) - f(0)| = |1 - c_1| \leq \varepsilon$ and $|g(\bar{t}) - f(\bar{t})| = |0 - c_1| \leq \varepsilon$. It follows that $\varepsilon \geq \inf\{|1-c|, |c| : c \in \mathbb{R}\} = 1/2$, and thus $g \notin \overline{\mathcal{E}}$. On the other hand given $\varepsilon > 0$, let P be the partition $0 < \varepsilon < 1$. We have that $L_P(g) = m_1\varepsilon + m_2(1-\varepsilon) = 0\varepsilon + 0(1-\varepsilon) = 0$ and $U_P(g) = M_1\varepsilon + M_2(1-\varepsilon) = 1\varepsilon + 0 = \varepsilon$, hence $\inf\{U_P(g) - L_P(g)\} = 0$ and g is Riemann integrable.

4. Given $N \in \mathbb{N}$, we have that $\mu(\cup_{n=1}^\infty E_n) \geq \mu(\cup_{n=1}^N E_n) = \sum_{n=1}^N \mu(E_n)$. Since N is arbitrary, it follows that $\mu(\cup_{n=1}^\infty E_n) \geq \sum_{n=1}^\infty \mu(E_n) = \sup\left\{\sum_{n=1}^N \mu(E_n) : N \in \mathbb{N}\right\}$

5. a) (i) $\mu^*(S) = \inf\{\sum \mu(E_n) : S \subseteq \bigcup E_n\}$

(ii) M is measurable with respect to μ^* if for all $S \subseteq X$,

$$\mu^*(S) = \mu^*(S \cap M) + \mu^*(S \cap M^c).$$

b) Let us suppose that $E \in \mathcal{E}$ and $S \subseteq X$. Then given $\varepsilon > 0$, choose $E_n \in \mathcal{R}$ such that

$$\sum \mu(E_n) \leq \mu^*(S) + \varepsilon.$$

Then since $S \cap E \subseteq \bigcup E_n \cap E$ and $S \cap E^c \subseteq \bigcup E_n \cap E^c$, it follows that

$$\begin{aligned} \mu^*(S) + \varepsilon \geq \sum \mu(E_n) &= \sum \mu(E_n \cap E) + \mu(E_n \cap E^c) \\ &\geq \mu^*(S \cap E) + \mu^*(S \cap E^c). \end{aligned}$$

Since $\varepsilon > 0$, it follows that $\mu^*(S) \geq \mu^*(S \cap E) + \mu^*(S \cap E^c)$. The reverse inequality follows from the fact that μ^* is automatically countably subadditive on any sets (a separate argument).