Handout Some remarks on lectures & ASSIGNMENTS

1. $\mathbb{R} \cong 2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$ $\cong$ means 1-1 correspondence

This is more precise than our result that $\mathbb{R} \not\cong \mathbb{N}$

i.e., $\mathbb{R}$ is not countable.

Proof: $2^{\mathbb{N}}$ is just another notation for $\mathcal{P}(\mathbb{N})$

a) We first note that $2^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$

i.e. there is a 1-1 corres between subsets of $\mathbb{N}$ and functions $a: \mathbb{N} \to \{0,1\}$, i.e. sequences $a_1, a_2, \dots$ where $a_k = a(k) \in \{0,1\}$. Simply use the 1-1 correspondence $S \to a = \mathbb{1}_S$.

b) Define $\theta: \{0,1\}^{\mathbb{N}} \to [0,1]$

$: (a_1, a_2, \dots) \mapsto \sum a_k 2^{-k}$

— we will write $\theta(a) = .a_1 a_2 a_3 \dots$

"binary point"

We have this is "not quite" 1-1 —

you can have $\theta(a) = \theta(b)$ (only) if you have the situation

$\theta(a) = .a_1 a_2 \dots a_{n-1} 1 0 \dots$

$\theta(b) = .a_1 a_2 \dots a_{n-1} 0 1 1 \dots$

"infinite 1-tail"

Let $S \subseteq \{0,1\}^{\mathbb{N}}$ be the sequences with 1-tails.

We have $\theta: \{0,1\}^{\mathbb{N}} \setminus S \to [0,1)$ is a 1-1 correspondence [NOTE: $\theta((1,1,1,\dots)) = 1$ is eliminated]

c) We just showed $\{0,1\}^{\mathbb{N}} \setminus S \cong [0,1)$.

We claim that $\{0,1\}^{\mathbb{N}} \cong [0,1)$.

($B \cap A = \emptyset$)

General fact: Say A is infinite and B is countable. Then $A \cong A \cup B$. "$\aleph + \aleph_0 = \aleph$"

d) We have that

$$\mathbb{R} \cong [0,1) \times \mathbb{N} \overset{*}{\cong} [0,1) \cong \{0,1\}^{\mathbb{N}} \cong 2^{\mathbb{N}}$$

* General fact: A infinite and B countable implies that $A \times \mathbb{N} \cong A$ "$\aleph \times \aleph_0 = \aleph$"

⇒ We'll accept "General facts" as well-known results from set theory.

2. The Cantor set

$$A_0 = [0,1]$$
$$A_1 = [0,1] \setminus (1/3, 2/3)$$
$$A_2 = A_1 \setminus [(1/9, 2/9) \cup (\tfrac{7}{9}, \tfrac{8}{9})]$$
$$\vdots$$
$$C = \bigcap A_n$$

Let $B_n = [0,1] \setminus A_n$. It consists of 2^{n-1} disjoint intervals each of length $\frac{1}{3^n}$, so

$$\lambda(B_n) = \frac{2^{n-1}}{3^n}$$

Let $B = \bigcup B_n$. We have that

geometric $a = \frac{1}{3}$, $r = 2/3$

$$\lambda(B) = \sum \lambda(B_n) = \sum_1^\infty \frac{2^{n-1}}{3^n} = \frac{1/3}{1-\frac{2}{3}} = 1$$

$$\lambda(C) = \lambda([0,1] \setminus B) = \lambda([0,1]) - \lambda(B) = 1 - 1 = 0$$

Define $\underline{\theta_0 : \{0,2\}^{\mathbb{N}} \longrightarrow [0,1]} : (a_n) \mapsto \sum_1^\infty \frac{a_n}{3^n} = \text{"}.a_1 a_2 a_3 \ldots\text{"}$

"ternary point"

$\underline{\theta_0 \text{ is one-to-one}}$. It suffices to show

$$\theta : \{0,1,2\}^{\mathbb{N}} \longrightarrow [0,1] : (a_n) \longmapsto \sum \frac{a_n}{3^n}$$

restricts to a 1-1 function on $\{0,2\}^{\mathbb{N}}$.

But if $a \in \{0,2\}^{\mathbb{N}}$ and $\theta(a) = \theta(b)$ $(a \neq b)$

we must have

beginning of 2-tail

$$\theta(a) = .a_1 a_2 \ldots a_{n-1}\, 0\, 2\, 2\, 2 \ldots$$
$$\theta(b) = .a_1 a_2 \ldots a_{n-1}\, 1\, 0\, 0\, 0 \ldots$$

i.e. we have $b \notin \{0,2\}^{\mathbb{N}}$.

θ_0 is a homeomorphism onto its image, i.e., θ_0 and θ_0^{-1} are continuous.

Proof: Since $\{0,1\}^{\mathbb{N}}$ is compact it suffices to show that θ_0 is continuous [$F \in \{0,1\}^{\mathbb{N}}$ closed $\Rightarrow$ F compact $\Rightarrow$ $\theta_0(F)$ compact $\Rightarrow$ $\theta_0(F)$ closed so F closed $\Rightarrow (\theta_0^{-1})^{-1}(F) = \theta_0(F)$ is closed].

We have to show that $a^{(k)} \to a$ implies that $\theta(a^{(k)}) \to \theta(a)$.

We have $a^{(k)} \to a \Rightarrow a^k(i) \to a(i)$

$\Rightarrow a^{(k)}(i)$ eventually $= a(i)$

Given n_0 we may assume

$k \geq k_0 \Rightarrow a^{(k)}(1) = a(1), \ldots a^{(k)}(n_0) = a(n_0)$

If $k \geq k_0$ then

$$|\theta(a^{(k)}) - \theta(a)| = \left|\sum_{m=n_0+1}^{\infty} (a^{(k)}(m) - a(m)) 3^{-m}\right|$$

$$\leq \sum_{n=n_0+1}^{\infty} 2 \cdot 3^{-n} = \frac{\frac{2}{3^{n_0+1}}}{1-\frac{1}{3}} = \frac{1}{3^{n_0}}$$

Given $\varepsilon > 0$ we may assume that $\frac{1}{3^{n_0}} < \varepsilon$.

<u>θ maps $\{0,2\}^{\mathbb{N}}$ onto C</u>

The points in A_1 are precisely those which have a ternary expansion

$.0a_2a_3\ldots$ or $.2a_2a_3\ldots$ (e.g. $\frac{1}{3} = .0222\ldots$)

(we <u>allow</u> infinite 2-tails). $\frac{2}{3} = .2000\ldots$

The points in A_2 are those of the form

$.a_1a_2a_3\ldots$ where $a_1, a_2 \in \{0,2\}$

We thus see that if $x \in C$, then $x \in A_n$ for all n, i.e.,

$x = .a_1a_2a_3\ldots \quad a_k \in \{0,2\}$

i.e., $x = \theta(a)$, and $C \subseteq \theta(\{0,2\}^{\mathbb{N}})$ On the other hand let

$$S = \{a \in \{0,2\}^{\mathbb{N}} \quad a = (a_1, a_2, \ldots, a_{n_0-1}, \overbrace{0,0,0\ldots}^{0\text{-tail}})$$

It is easy to see that $\theta(a) \in C$ since in fact $\theta(a)$ is an <u>end point</u> of one of the intervals in A_{n_0} and it is never eliminated when one deletes subsequent middle third sets. Thus $\theta(S) \subseteq C$.

But we have that $\bar{S} = \{0,2\}^{\mathbb{N}}$ (<u>why?</u>)

Since θ is continuous, ...

3. The proof that $\lambda : \mathcal{R}(\mathcal{J}) \to [0,\infty)$ is countably additive.

Since we already know that λ is finitely additive, it suffices to prove

$$E \in \mathcal{R}(\mathcal{J}),\ E_n \in \mathcal{R}(\mathcal{J}),\ E \subseteq \bigcup_{n=1}^{\infty} E_n$$

$$\Rightarrow \lambda(E) \le \sum_{n=1}^{\infty} \lambda(E_n).$$

Case 1: $E = [a,b)$ $E_n = [a_n, b_n)$

See Class Notes.

Case 2: $E = I_1 \sqcup \cdots \sqcup I_p$ $E_n = J_n$

We have

$$I_1 \sqcup \cdots \sqcup I_p \subseteq \bigcup J_n$$

$$\Rightarrow \quad I_i \subseteq \bigcup I_i \cap J_n$$

$$\Rightarrow \quad \lambda(I_i) \le \sum_{n=1}^{\infty} \lambda(I_i \cap J_n) \qquad \text{(by case 1)}$$

$$\begin{aligned} \Rightarrow \quad \lambda(E) &= \sum_1^p \lambda(I_i) \\ &\le \sum_{i=1}^{p} \sum_{n=1}^{\infty} \lambda(I_i \cap J_n) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{p} \lambda(I_i \cap J_n) \\ &= \sum_{n=1}^{\infty} \lambda(E \cap J_n) \end{aligned}$$

Since $E \cap J_n = \bigsqcup_{i=1}^{p} I_i \cap J_n$ (λ is finitely additive)

Case 3 $E \in \mathcal{R}(\mathcal{J})$, $E_n = J_{n1} \sqcup \cdots \sqcup J_{n g_n} \in \mathcal{R}(\mathcal{J})$

$$E \subseteq \bigcup E_n \Rightarrow E \subseteq \bigcup_{n,j} J_{nj}$$

$$\begin{aligned} \Rightarrow \lambda(E) &\le \sum_{i,j} \lambda(J_{nj}) \qquad \text{(by case 2)} \\ &= \sum_i \lambda(E_n) \end{aligned}$$

because $E_n = \bigsqcup_{j=1}^{g_n} J_{nj}$ and λ is finitely additive.