Answers to Quiz #1.

1. Since $\varphi: S \to T$ is surjective, for each $t \in T$ we have $E_t = \varphi^{-1}(t)$ is non-empty. Let ψ be a choice function for $\{E_t\}_{t \in T}$, i.e., $\psi : T \to \bigcup E_t$ and $\psi(t) \in E_t \subseteq S$. Then

$$\psi : T \to S$$

is 1-1 since $t_1 \neq t_2$ implies $E_{t_1} = \varphi^{-1}(t_1)$ and $E_{t_2} = \varphi^{-1}(t_2)$ are disjoint, so $\psi(t_1)$ and $\psi(t_2)$ are in disjoint sets.

2. a) Yes: Say that t_n is a sequence in T. Let $s_n \in f^{-1}(t_n)$. Since S is compact we may choose a subsequence $s_{n_k} \to s \in S$. Then

$$t_{n_k} = f(s_{n_k}) \to f(s)$$

since f is continuous, i.e., t_{n_k} converges.

b) No: Consider the continuous surjection

$$f : [0,\infty) \to [0,1) : t \mapsto \frac{t}{1+t}$$

$[0,1)$ is not complete because, e.g., it is not closed in $[0,1]$. $[0,\infty)$ is a closed subset of $\mathbb{R}$ so it is complete.

3. f is cont. on $[0,1] \Rightarrow f$ is uniformly continuous on $[0,1]$.
If $s, t \geq 1$, then

$$|f(s) - f(t)| = |\sqrt{s} - \sqrt{t}|$$
$$= \frac{|s-t|}{\sqrt{s} + \sqrt{t}} \leq \frac{|s-t|}{2}$$

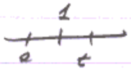
Given $\varepsilon > 0$, let $\delta = 2\varepsilon$. Then

$$|s-t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$$

so $f : [1,\infty) \to \mathbb{R}$ is unif cont.

Given $\varepsilon > 0$, choose δ so that

$$|s-t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon/2$$



if $s,t \in [0,1]$ or if $s,t \in [1,\infty)$. Then $s \in [0,1]$ and $t \in [1,\infty)$ and $|s-t| < \delta \Rightarrow |s-1| < \delta$ and $|t-1| < \delta \Rightarrow$

$$|f(s)-f(t)| \le |f(s)-f(1)| + |f(1)-f(t)| < \varepsilon.$$

So it's clear that $\forall s,t \in \mathbb{R}$,

$$|s-t|<\delta \Rightarrow |f(s)-f(t)|<\varepsilon.$$

4. Remember that $P \Rightarrow Q$ is equivalent to $\sim Q \Rightarrow \sim P$.

Say that $\lim\limits_{\substack{x\to a \\ x\ne a}} f(x) \ne f(a)$. Then

$$\sim\left[\forall \varepsilon>0, \exists \delta>0\ \forall x,\ d(x,a)<\delta \Rightarrow d(f(x),f(a))<\varepsilon\right]$$

i.e., $\exists \varepsilon>0,\ \forall \delta>0,\ \exists x,\ d(x,a)<\delta$ and $d(f(x),f(a)) \ge \varepsilon$

Fix such an $\varepsilon>0$ and choose x_n such that $d(x_n,a)<\frac{1}{n}$ and $d(f(x_n),f(a)) \ge \varepsilon$. Then $x_n \to a$ but $f(x_n) \not\to f(a)$.

5. Say f_n is Cauchy. Then for each $\omega \in \Omega$

$$|f_n(\omega)-f_m(\omega)| \le \|f_n-f_m\|_\infty$$

Hence $f_n(\omega)$ is Cauchy. Since $\mathbb{R}$ is complete we may define $f(\omega)=\lim\limits_{n\to\infty} f(\omega)$.

a) f is bounded: Note that $\{\|f_n\|_\infty\}$ is Cauchy since

$$|\,\|f_n\|_\infty-\|f_m\|_\infty\,| \le \|f_n-f_m\|_\infty$$

Thus $\{\|f_n\|_\infty\}$ is a bounded sequence (of reals): let $\|f_n\|_\infty \le M$ for all n. Then for each ω

$$|f_n(\omega)| \le \|f_n\|_\infty \le M$$

Hence $\underbrace{|f(\omega)| = \lim |f_n(\omega)| \le M}$ ($|\cdot|$ is cont.)

and since true for all ω, $\|f\|_\infty \le M$

b) Given $\varepsilon>0$ choose N such that

$$m,n \ge N \Rightarrow \|f_m-f_n\|_\infty<\varepsilon.$$

Then for each ω,

$$|f_m(\omega)-f_n(\omega)|<\varepsilon$$

$\Rightarrow$ $|f(\omega)-f_n(\omega)| \le \varepsilon$ (take limit on m)

$\Rightarrow$ $\|f-f_n\|_\infty \le \varepsilon$ for $n \ge N$