Handout 3 Review of Continuity

Say $f: M \to N$, $a \in M$.

<u>Def:</u> f is continuous at a if

$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon$

i.e., $\forall \varepsilon > 0, \exists \delta > 0 : \quad f(B_\delta(a)) \subseteq B_\varepsilon(f(a))$

We also write $\lim\limits_{\substack{x \to a \\ x \neq a}} f(x) = f(a)$

<u>Prop:</u> f is cont at $a \iff x_n \to a \Rightarrow f(x_n) \to f(a)$.

<u>Pf.</u> $\Rightarrow$) Given $\varepsilon > 0$, choose $\delta > 0$ such that

$d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon$

Given $x_n \to a$, choose N so that $n \geq N \Rightarrow d(x_n, a) < \delta$. Then $n \geq N \Rightarrow d(f(x_n), f(a)) < \varepsilon$.

$\Leftarrow$) Say f is not cont at a. Then

$\exists \varepsilon > 0, \forall \delta > 0, \quad f(B_\delta(a)) \not\subseteq B_\varepsilon(f(a))$.

Fix such an $\varepsilon > 0$, and for each $n \in \mathbb{N}$ choose $x_n \in B_{\frac{1}{n}}(a)$ with $f(x_n) \notin B_\varepsilon(f(a))$, i.e., $d(f(x_n), f(a)) \geq \varepsilon$.

Since $d(x_n, a) < \frac{1}{n}$, $x_n \to a$, but $f(x_n) \not\to f(a)$.

<u>Prop:</u> Say $f: M \to N$. Then the following are equivalent

(1) f is continuous (i.e., it is continuous at all $a \in M$)

(2) G open in $N \Rightarrow f^{-1}(G)$ is open in M

(3) F closed in $N \Rightarrow f^{-1}(F)$ is closed in M.

<u>Pf</u> For (1)$\iff$(2) see 0.23.

<u>Note</u> $f: M \to N$ is <u>cont</u>:

$\forall a \in M, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in M, d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon$

$f: M \to N$ is <u>uniformly</u> <u>cont</u>:

$\forall \varepsilon > 0, \exists \delta > 0 : \forall a \in M, \forall x \in M, d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon$

Prop: M compact, $f: M \to N$ continuous $\Rightarrow$ f is uniformly continuous

Proof: Say f is not unif. cont. Then

$$\exists \varepsilon > 0,\ \forall \delta > 0 \ \sim [\forall x, \forall y,\ d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon]$$

Fix such an $\varepsilon > 0$, and for each $n \in \mathbb{N}$ choose x_n, y_n such that

$$d(x_n, y_n) < \tfrac{1}{n} \text{ but } d(f(x_n), f(y_n)) \geq \varepsilon.$$

Since M is compact we may assume $x_{n_k} \to a$

Since $(y_{n_k})_{k \in \mathbb{N}}$ is a sequence in M we

may assume $y_{n_{k_j}} \to b$. We necessarily have $x_{n_{k_j}} \to a$.

Changing notation, i.e., replacing $x_{n_{k_j}}$ by x_j

and $y_{n_{k_j}}$ by y_j we have

$$x_j \to a \text{ and } y_j \to b$$

and thus $f(x_j) \to f(a)$ and $f(y_j) \to f(b)$

Since $d(x_j, y_j) \to d(a,b)$ and $d(x_j, y_j) \to 0$,

we have $d(a,b) = 0 \Rightarrow a = b$. Thus

$$d(f(x_j), f(y_j)) \to d(f(a), f(a)) = 0$$

contradicting the fact that for all j,

$$d(f(x_j), f(y_j)) \geq \varepsilon.$$