Math 275a   Handout 2: Axiom of Choice

Def. $(a,b) = \{\{a\}, \{a,b\}\}$

Def: $A\times B = \{(a,b): a\in A,\ b\in B\}$ e.g. $A\times\emptyset = \emptyset\times B = \emptyset$

Def: A function $f: A \to B$ is a subset $f \subseteq A\times B$ such that
(1) $\forall a\in A,\ \exists b\in B: (a,b)\in f$
(2) $\forall a,\ \forall b_1\in B,\ \forall b_2\in B,\ (a,b_1)\in f$ and $(a,b_2)\in f \Rightarrow b_1 = b_2$.

We write $f(a) = b$ if $(a,b)\in f$, or $f_a = b$, or $f: a\mapsto b$.

COMPLETE:

f is one-to-one if ...
f is onto if ...
f is a bijection if ...

Given a set S, a sequence $a_1, a_2, \ldots$ in S is a function $a: \mathbb{N} \to S$ (recall $a_n = a(n)$). We write $a = (a_n)_{n\in\mathbb{N}}$

Eg. A sequence of subsets A_n of S is a map
$A: \mathbb{N} \to \mathcal{P}(S)$ $A = (A(n))_{n\in\mathbb{N}}$

More generally: Given an arbitrary set I and a map $a: I \to S$ we may regard $\{a_\alpha = a(\alpha): \alpha\in I\}$ as an "indexed family of elements in S".
We also write $a = (a_\alpha)_{\alpha\in I}$

Given an indexed family $\{A_\alpha : \alpha\in I\}$, $A_\alpha \subseteq S$ we let

πA_α = set of all functions $a: I \to S$ such that $a_\alpha \in A_\alpha$.

We call an element $a\in \pi A_\alpha$ a choice function for $\{A_\alpha : \alpha\in I\}$
If $A_\alpha = B$ for all $\alpha\in I$ we let
$$B^I = \prod_{\alpha\in I} A_\alpha$$

Examples $\prod_{i\in\{1,2\}} A_i = \{a : \{1,2\} \to A_1 \cup A_2 : a_1 \in A_1, a_2 \in A_2\}$
$= A_1 \times A_2$

$\mathbb{R}^{\mathbb{R}}$ = all fns $f : \mathbb{R} \to \mathbb{R}$

e.g., $f(x) = c$ all x or $f(x) = x^2$, etc.

Given subsets $A_\alpha \subseteq \mathbb{N}$ $(\alpha \in I)$ with $A_\alpha \neq \emptyset$ $\forall \alpha$ we can explicitly describe an element
$$a \in \prod_{\alpha\in I} A_\alpha$$

– e.g. let $a(\alpha)$ = least element of A_α (remember $S_\alpha \subseteq \mathbb{N}$)

"NO ONE KNOWS HOW TO DO THIS" for $A_\alpha \subseteq \mathbb{R}, A_\alpha \neq \emptyset$ unless either a) sets A_α are "special"

E.g. if A_α are bounded closed sets, we can let
$$a(\alpha) = \inf A_\alpha$$

or b) I is finite

$I = \{1,2,3\}$ "let $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$ . . ."

– if I is infinite you would need infinitely many sentences

or c) $I = \mathbb{N}$

"Use induction": Let $a_1 \in A_1$. Having chosen $a_n \in A_n$ let $a_{n+1} \in A_{n+1}$. Then
$$T = \{n \in \mathbb{N} : a_n \text{ is defined}\}.$$
is equal to $\mathbb{N}$ by induction.

NOTE: Induction lets us avoid writing down infinitely many sentences.

$\Rightarrow$ Mathematics becomes a lot more elegant if you ignore these problems by adopting

Axiom of Choice: Given a family of sets $\{A_\alpha\}_{\alpha\in I}$ with $A_\alpha \subseteq S$, $\forall \alpha \in I$, $A_\alpha \neq \emptyset$, there exists a choice function $a : I \to S$ with $a(\alpha) \in A_\alpha$.