Handout 10: A cleaner proof for Cor. 2.32

From the book: $f_n \to f$ <u>in measure</u> if for all $\varepsilon > 0$

$$\mu\{x : |f_n(x) - f(x)| \geq \varepsilon\} \to 0 \text{ as } n \to \infty$$

<u>Prop:</u> If $\|f_n - f\|_1 \to 0$, then $f_n \to f$ in measure. (2.29)

<u>Theorem</u>: If $f_n \to f$ in measure, then there is a subsequence f_{n_k} such that $f_{n_k}(x) \to f(x)$ for almost all x.

<u>Proof:</u> Choose n_1 such that

$$\mu(\{x : |f_{n_1}(x) - f(x)| \geq 1\}) < \frac{1}{2}$$

Having chosen $n_1, \dots, n_k$ choose n_{k+1} so that

$$\mu(\{x : |f_{n_{k+1}}(x) - f(x)| \geq \tfrac{1}{k+1}\}) < \frac{1}{2^{k+1}}$$

Let $$A_j = \bigcup_{k=j}^{\infty} \{x : |f - f_{n_k}| \geq \tfrac{1}{k}\}.$$

We note that $\mu(A_j) \leq \sum_{k=j}^{\infty} \mu(\{x : |f - f_{n_k}| \geq \frac{1}{k}\}) < \sum_{k=j}^{\infty} 2^{-k} = 2^{-j+1}$

hence $\mu(A_j) \to 0$. Let $B = \bigcap_{j=1}^{\infty} A_j$. Since $A_1 \supseteq A_2 \supseteq \dots$

and $\mu(A_1) < \infty$, we have $\mu(B) = \lim \mu(A_j) = 0$.

Say that $x_0 \notin B$, i.e., $x_0 \in \bigcup_{j=1}^{\infty} A_j^c$. Choose j

such that $x_0 \in A_j^c = \bigcap_{k=j}^{\infty} \{x : |f - f_{n_k}| < \frac{1}{k}\}$.

We see that $k \geq j \Rightarrow |f(x_0) - f_{n_k}(x_0)| < \frac{1}{k}$

and thus $f_{n_k}(x_0) \to f(x_0)$. Since $\mu(B) = 0$ we are done.

QED.