Assignment 2

Math 245a, Fall 2002

Due: Mon, Oct. 14

1. Prove that if $(\mathcal{M}, d)$ is a metric space, then the map $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is continuous.

   Suggestion: Show that if $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$.

2. Say that $(\mathcal{M}, d)$ is a metric space and that $F_n \subseteq \mathcal{M}$ is closed. Does it follow that $\bigcup F_n$ is closed? Prove your assertion.

3. Suppose that $(\mathcal{M}, d)$ and $(\mathcal{N}, \rho)$ are homeomorphic metric spaces and that $\mathcal{M}$ is complete. Does it follow that $\mathcal{N}$ is complete? Prove your assertion.

4. Find a closed and bounded subset of $\mathbb{Q}$ which is not compact.

5. Say that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$ and that $\mathcal{A}$ is dense in $\mathcal{B}$, and $\mathcal{B}$ is dense in $\mathcal{M}$. Carefully show that $\mathcal{A}$ is dense in $\mathcal{M}$.

6. Give an example of a bounded and complete metric space $\mathcal{M}$ which is not compact.

7. a. Show that if K is compact and F is closed in $\mathcal{M}$, and $K \bigcap F = \varnothing$, then there is an $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for all $x \in K$, $y \in F$.

   b. Show (a) is false if you assume K and F are closed in $\mathcal{M}$.

8. a. Show that if $\mathcal{M}$ and $\mathcal{N}$ are metric spaces, and $f : \mathcal{M} \to \mathcal{N}$ is continuous, then f is a closed subset of $\mathcal{M} \times \mathcal{N}$.

   b. Show that the converse of the above statement is false.