

Fractional Linear Transformations ("FLT's")

You should be able to prove these and answer "why?" questions

General FLT: $f(z) = \frac{az + b}{cz + d}$, [Affine $f(z) = az + b$]
 $c \neq 0$, i.e., $c=0$

Special cases

a) Translation $f(z) = z + c = T_c(z)$

b) Dilation $f(z) = cz = M_c(z)$

c) Inversion $f(z) = \frac{1}{z} = I(z)$

Theorem Every FLT is a composition of special FLT's

Pf: a) Affine $f(z) = az + b$ is the composition

$$z \mapsto az \mapsto az + b$$

as Non-affine, i.e., $c \neq 0$. We may assume $c = 1$

(simply divide numerator & denominator by c). Thus

$$f(z) = \frac{az + b}{cz + d} = \frac{az + ad}{cz + d} + \frac{-ad + b}{cz + d}$$

$$= a + \frac{-ad + b}{cz + d}$$

hence f is given as a composition

$$\begin{array}{ccccccc} z & \mapsto & z+d & \mapsto & \frac{1}{z+d} & \mapsto & -ad+b \\ & & T_d & & I & & M_{(-ad+b)} \\ & & & & & & T_a \end{array}$$

Theorem: f fractional Linear $\Rightarrow f$ sends circles and lines to circles and lines

Pf: [Use \mathbb{R}^2 notation]

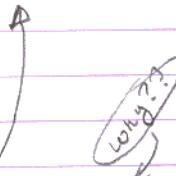
General EQUATION of circle, line:

$$(1) \Gamma: Ax^2 + Ay^2 + Bx + Cy + D = 0$$

Suffices to show each special FLT f has that property.

a) Let $c = a+bi = (a, bi)$. Then

$$T_c(x, y) = (a, bi) + (x, y) = (a+x, bi+y)$$



$$\text{L} \rightarrow c = r > 0 \quad M_r(x, y) = r(x, y) = (rx, ry)$$

$$\text{L} \rightarrow c = e^{i\theta} \quad M_{e^{i\theta}}(x, y) = (\cos \theta, \sin \theta)(x, y)$$

$$= (\cos \theta x - \sin \theta y, \cos \theta y + \sin \theta x)$$

Note: $\sqrt{c} = r e^{i\theta}$, then $M_c = M_r M_{e^{i\theta}}$ why??)

$$\text{c)} \quad I(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$I(x, y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$$

a) The equation for $T_c(\Gamma)$

$$(u, v) \in T_c(\Gamma) \Leftrightarrow T_c(u, v) \in \Gamma$$

$$\Leftrightarrow (u-a, v-b) \in \Gamma$$

$$\Leftrightarrow A(u-a)^2 + A(v-b)^2 + B(u-a) + C(v-b) + D = 0$$

$$\Leftrightarrow Au^2 + Av^2 + (-2a+B)u + (-2b+C)v + Aa^2 + Bb^2 - Ba - Cb + D = 0$$

which again has the form (+)

b) The equation for $M_c(\Gamma)$

L) say that $c = r > 0$.

$$(u, v) \in M_r(\Gamma) \Leftrightarrow M_r(u, v) \in \Gamma \Leftrightarrow (r^2 u, r^2 v) \in \Gamma$$

$$\Leftrightarrow A(r^2 u)^2 + A(r^2 v)^2 + B(r^2 u) + C(r^2 v) + D = 0$$

$$\Leftrightarrow (Ar^2)^2 u^2 + (Ar^2)^2 v^2 + (Br^2)u + (Cr^2)v + D = 0$$

R) say that $c = e^{i\theta}$. Note $e^{-i\theta} = e^{-i\theta}$

$$(u, v) \in M_{e^{i\theta}}(\Gamma) \Leftrightarrow M_{e^{-i\theta}}(u, v) \in \Gamma$$

$$\Leftrightarrow (\cos(-\theta)u - \sin(-\theta)v, \cos(-\theta)v + \sin(-\theta)u) \in \Gamma$$

$$\Leftrightarrow A(\cos^2 \theta u^2 + 2\cos \theta \sin \theta uv + \sin^2 \theta v^2)$$

$$+ A\cos^2 \theta v^2 - 2\cos \theta \sin \theta uv - \sin^2 \theta u^2)$$

$$+ B(u^2 + v^2) + C(\cos \theta u - \sin \theta v) + D = 0$$

$$\Leftrightarrow Au^2 + Bv^2 + (B\cos \theta - C\sin \theta)u + (B\sin \theta + C\cos \theta)v + D = 0$$

c) The equation for $I(\Gamma)$. Note $I^{-1} \subseteq \Gamma$ (why??)

$$(u, v) \in I(\Gamma) \Leftrightarrow I(u, v) \in \Gamma \Leftrightarrow \left(\frac{u}{u^2+v^2}, \frac{-v}{u^2+v^2} \right) \in \Gamma$$

$$\Leftrightarrow \frac{Au^2}{(u^2+v^2)^2} + \frac{Av^2}{(u^2+v^2)^2} + Bu - \frac{Cv}{u^2+v^2} + D = 0$$

MULTIPLY by u^2+v^2 $\Leftrightarrow A + Bu - Cv + D(u^2+v^2) = 0$