Fractional Linear Transformations ("FLT's")

You should be able to prove these and answer "why??" questions

General FLT: $f(z) = \dfrac{az+b}{cz+d}$, [Affine $f(z) = az+b$ i.e., $c=0$]

Special cases

a) Translation $f(z) = z + c = T_c(z)$

b) Dilation $f(z) = cz = M_c(z)$

c) Inversion $f(z) = \frac{1}{z} = I(z)$

Theorem Every FLT is a composition of special FLT's

Pf: a) Affine $f(z) = az+b$ is the composition

$$z \longmapsto az \longmapsto az+b$$

b) Non-affine, i.e., $c \neq 0$. We may assume $c=1$ (simply divide numerator & denominator by c). Thus

$$f(z) = \frac{az+b}{z+d} = \frac{az+ad}{z+d} + \frac{-ad+b}{z+d}$$

$$= a + \frac{-ad+b}{z+d}$$

Hence f is given as a composition

$$z \xrightarrow[T_d]{} z+d \xrightarrow[I]{} \frac{1}{z+d} \xrightarrow[M_{(-ad+b)}]{} \frac{-ad+b}{z+d} \xrightarrow[T_a]{} a + \frac{-ad+b}{z+d}$$

Theorem: f fractional Linear $\Rightarrow$ f sends circles and lines to circles and lines

Pf: $\boxed{\text{Let's use } \mathbb{R}^2 \text{ notation}}$

General Equation of circle, line:

(*) Γ: $Ax^2 + Ay^2 + Bx + Cy + D = 0$

Suffices to show each special FLT f has that property

why??

a) Let $c = a + ib = (a, b)$. Then

$$T_c(x,y) = (a,b) + (x,y) = (a+x, b+y)$$

b_1) $c = r > 0$ $\quad M_r(x,y) = r(x,y) = (rx, ry)$

b_2) $c = e^{i\theta}$ $\quad M_{e^{i\theta}}(x,y) = (\cos\theta, \sin\theta)(x,y)$

$$= (\cos\theta\, x - \sin\theta\, y,\ \cos\theta\, y + \sin\theta\, x)$$

Note: If $c = re^{i\theta}$, then $M_c = M_r M_{e^{i\theta}} \leftarrow$ (why??) :)

c) $I(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$$I(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

a) $\boxed{\text{The equation for } T_c(\Gamma)}$

$$(u,v) \in T_c(\Gamma) \Leftrightarrow T_{-c}(u,v) \in \Gamma$$
$$\Leftrightarrow (u-a, v-b) \in \Gamma$$
$$\Leftrightarrow A(u-a)^2 + A(v-b)^2 + B(u-a) + C(v-b) + D = 0$$
$$\Leftrightarrow Au^2 + Av^2 + (-2a + B)u + (-2b + C)v$$
$$+ Aa^2 + Bb^2 - Ba - Cb + D = 0$$

which again has the form (*)

b) $\boxed{\text{The equation for } M_c(\Gamma)}$

b_1) Say that $c = r > 0$.

$$(u,v) \in M_r(\Gamma) \Leftrightarrow M_{r^{-1}}(u,v) \in \Gamma \Leftrightarrow (r^{-1}u, r^{-1}v) \in \Gamma$$
$$\Leftrightarrow A(r^{-1}u)^2 + A(r^{-1}v)^2 + B(r^{-1}u) + C(r^{-1}v) + D = 0$$
$$\Leftrightarrow (Ar^{-2})u^2 + (Ar^{-2})v^2 + (Br^{-1})u + (Cr^{-1})v + D = 0$$

b_2) Say that $c = e^{i\theta}$. Note $c^{-1} = e^{-i\theta}$

$$(u,v) \in M_{e^{i\theta}}(\Gamma) \Leftrightarrow M_{e^{-i\theta}}(u,v) \in \Gamma$$
$$\Leftrightarrow (\cos(-\theta)u - \sin(-\theta)v,\ \cos(-\theta)v + \sin(-\theta)u) \in \Gamma$$
$$\Leftrightarrow A(\cos^2\theta\, u^2 + 2\cos\theta\sin\theta\, uv + \sin^2\theta\, v^2)$$
$$+ A(\cos^2\theta\, v^2 - 2\cos\theta\sin\theta\, uv + \sin^2\theta\, u^2)$$
$$+ B(\cos\theta + \sin\theta\, v) + C(\cos\theta\, v - \sin\theta\, u) + D = 0$$
$$\Leftrightarrow Au^2 + Bv^2 + (B\cos\theta - C\sin\theta)u + (B\sin\theta + C\cos\theta)v + D = 0$$

NOTE $I^{-1} = I$

c) $\boxed{\text{The equation for } I(\Gamma)}$ NOTE $I^{-1} = I$ (why??)

$$(u,v) \in I(\Gamma) \Leftrightarrow I(u,v) \in \Gamma \Leftrightarrow \left(\frac{u}{u^2+v^2}, -\frac{v}{u^2+v^2}\right) \in \Gamma$$
$$\Leftrightarrow \frac{Au^2}{(u^2+v^2)^2} + \frac{Av^2}{(u^2+v^2)^2} + \frac{Bu}{u^2+v^2} - \frac{Cv}{u^2+v^2} + D = 0$$

MULTIPLY by u^2+v^2 $\Leftrightarrow A + Bu - Cv + D(u^2+v^2) = 0$