Practice Final Answers

1 a) Let $\log z = \log|z| + i \arg z$ where $\arg z$ is the continuous "branch" of the many valued "angle" function in the given region — we may assume that $\arg z = 0$ on the ray $(0,\infty)$

b) $u = \log|z| = \frac{1}{2}\log(x^2+y^2)$

$v = \tan^{-1}\frac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)$$

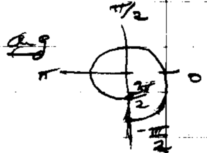
$$= -\frac{y}{x^2+y^2} = -\frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial y}\frac{y}{x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{y} = \frac{y}{x^2+y^2} = \frac{\partial u}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}$$

c) Let $\gamma: [a,b] \to \mathbb{C}$ be the indicated curve. Then

$$\int_\gamma \frac{dz}{z} = \left[\log z\right]_{\gamma(a)}^{\gamma(b)} = \log(-i) - \log(-2i)$$



$$= \log|-i| + i\arg(-i) - \log|-2i| + i\arg(-2i)$$

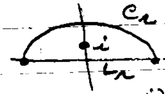
$$= \overbrace{\log 1}^{0} + i\left(\frac{3\pi}{2}\right) - \left(\log 2 + i\left(-\frac{\pi}{2}\right)\right)$$

$$= -\log 2 + i\left[\frac{3\pi}{2} + \frac{\pi}{2}\right] = -\log 2 + 2\pi i$$

2. Say $|f(z)| \le K$ for all z. Then fix z and let $r \to \infty$

$$|f'(z)| \le ML = \frac{K}{2\pi}\cdot\frac{1}{r^2}\,2\pi r = \frac{K}{r} \to 0$$

$\Rightarrow f'(z) = 0 \Rightarrow f$ constant

3.  C_R $\quad \dfrac{e^{i2z}}{1+z^2} \quad \Gamma_R = L_R + C_R$

$$\text{Res}_{z=i}\, \frac{e^{i2z}}{1+z^2} = \frac{e^{i(2i)}}{2i} = \frac{e^{-2}}{2i} \quad \left[\text{use } \text{Res}[f(z); z_0] = \frac{f(z_0)}{g'(z_0)}\right]$$

for simple pole

$$\int_{\Gamma_R} \frac{e^{i2z}}{1+z^2}\,dz = 2\pi i \sum[\text{Res } f, z_i] = 2\pi i[\text{Res } f, i]$$

$$= 2\pi i \times \frac{e^{-2}}{2i} = \frac{\pi}{e^2}$$

$$\frac{\pi}{e^2} = \int_{\Gamma_R} = \int_{L_R} + \int_{C_R} = \underbrace{\int_{-R}^{R} \frac{e^{i2x}}{1+x^2}\,dx}_{\downarrow\ \int_{-\infty}^{\infty} \frac{e^{i2x}}{1+x^2}\,dx} + \underbrace{\int_{C_R} \frac{e^{i2z}}{1+z^2}\,dz}_{\downarrow (*)\ 0}$$

(*) because $\left|\int_{C_R} \frac{e^{i2z}}{1+z^2}\,dz\right| \le ML$

$z \in C_R$: $\left|\dfrac{e^{i2z}}{1+z^2}\right| = \dfrac{|e^{i(2x+2yi)}|}{|1+z^2|} \le \dfrac{|e^{i2x}||e^{-2y}|}{R^2-1} \le \dfrac{1}{R^2-1}$

since $|e^{i2x}| = 1$, $|z^2+1| = |z^2-(-1)| \ge ||z|^2 - |1|| = R^2-1$

and $y \ge 0 \Rightarrow e^{-2y} \le 1$

4. Let $f = g^{-1}$ where g maps $(-i, \infty, 1)$ to $(0, 1, \infty)$

$$w = g(z) = \frac{z-(-i)}{z-1} \cdot \frac{\infty - 1}{\infty-(-i)} = \frac{z+i}{z-1}$$

Solve for z

$zw - w = z + i$

$z(w-1) = w + i$

$f(w) = z = \dfrac{w+i}{w-1}$.

5. $\dfrac{e^z}{z^2(\pi - z)}$ ~~double~~ pole at $z=0$

simple pole at $z = \pi \Rightarrow$ use f/g formula

$$\text{Res}\left[\frac{e^z}{\pi z^2 - z^3}, \pi\right] = \left[\frac{e^z}{2\pi z - 3z^2}\right]_{z=\pi} = \frac{e^{\pi}}{2\pi^2 - 3\pi^2} = \frac{-e^{\pi}}{\pi^2}$$

$$f(z)=\frac{e^z}{z^2(\pi-z)}=\frac{a_{-2}}{z^2}+\frac{a_{-1}}{z}+a_0+\cdots$$

Let $h(z)=z^2\left[\frac{e^z}{z^2(\pi-z)}\right]=a_{-2}+a_{-1}z+a_0z^2+\cdots$

$$\Rightarrow \operatorname{Res}[f,0]=a_{-1}=h'(0)$$

$$h'(z)=\left[\frac{e^z}{(\pi-z)}\right]'=(\pi-z)e^z-e^z(\pi-z)'=(\pi-z)e^z+e^z$$
$$=(\pi-z+1)e^z$$

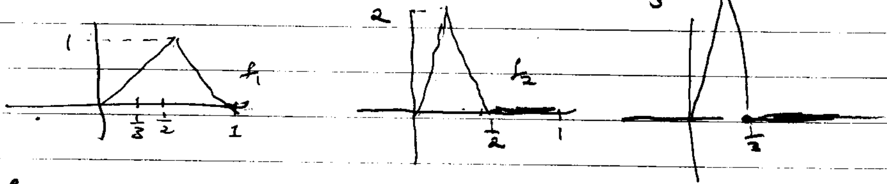
$[\operatorname{Res} f;0]=h'(0)=(\pi+1)e^0=\pi+1.$

6. $z^2\sin\frac{1}{z}=z^2\left[\frac{1}{z}-\frac{1}{3!}\frac{1}{z^3}+\frac{1}{5!}\frac{1}{z^5}-\cdots\right]$

$$=z-\underbrace{\frac{1}{3!}\frac{1}{z}+\frac{1}{5!}\frac{1}{z^3}-\cdots}_{\text{singular part}}$$

7. Let $v(z)$ be a harmonic conjugate of $u(z)$, so that $f(z)=u+iv$ is analytic. We have
$$F(z)=e^{-f(z)}$$
is analytic on all of $\mathbb{C}$, and if $u(z)\geq K$,
$$|F(z)|=|e^{-u(z)+iv(z)}|=|e^{-u(z)}|\leq e^{-K}$$
It follows that $F(z)$ is constant, hence so is
$$|F(z)|=e^{-u(z)}$$
But $x\mapsto e^{-x}$ is one-to-one on $\mathbb{R}$ since it is strictly decreasing (the derivative is <0). Thus $u(z)$ is constant.

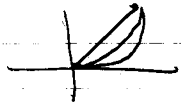
8.



$f_n(x)\to 0$ for all $x\in[0,1]$

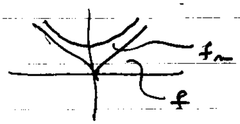
$\int_0^1 f_n(x)\,dx=$ area of triangle $=\frac{1}{2}\frac{1}{n}\cdot n=\frac{1}{2}$

$\int_0^1 f_n(x)\,dx\to\frac{1}{2}\qquad \int_0^1 f(x)\,dx=\int_0^1 0\,dx=0$

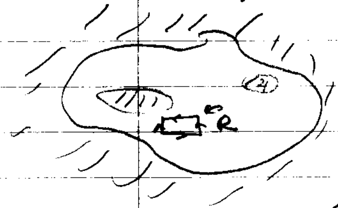
b)  $f_n(x) = x^n \quad 0 \le x \le 1 \quad n = 1, 2, \ldots$

 $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

If $x < 1$, $x^n \to 0$; if $x = 1$ $x^n = 1 \to 1$.

c) 

9. By Morera's theorem, $f(z)$ is analytic $\Leftrightarrow \int_{\partial R} f(z)\,dz = 0$ for all closed rectangles contained in G (and sides parallel to x, y axes). Thus f_n analytic $\Rightarrow$



$$0 = \int_{\partial R} f_n(z)\,dz \to \int_{\partial R} f(z)\,dz$$

since $f_n \to f$ uniformly, $\Rightarrow f$ is analytic.

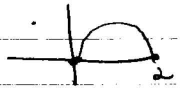
Given $z \in G$ let D_r be the closed disk center z and radius r. Then

$$(f_n^{(k)} - f^{(k)})(z) = \frac{n!}{2\pi i} \int_{\partial D_r} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

$$|f_n^{(k)}(z) - f^{(k)}(z)| \le \frac{n!}{2\pi} \frac{\varepsilon_n}{r^{n+1}} \cdot 2\pi r = \frac{n!\,\varepsilon_n}{r^n} \to 0$$

where $|f_n(\xi) - f(\xi)| \le \varepsilon_n \to 0$.

__Note__: Book shows $f_n^{(k)} \to f^{(k)}$ uniformly on proper subdisks of a disk.

10. 

γ_1: $x = t \quad 0 \le t \le 2 \quad \Rightarrow dz = dt$
$y = 0$

$$\int_{\gamma_1} \bar{z}\,dz = \int_0^2 (x - iy)\,dt = \int_0^2 t\,dt = \frac{t^2}{2}\Big]_0^2 = 2$$

γ_2: $z = 1 + e^{it} \quad 0 \le t \le \pi \quad dz = ie^{it}dt \quad \bar{z} = 1 + e^{-it}$

$$\int_{\gamma_2} \bar{z}\,dz = \int_0^{\pi} (1 + e^{-it})\, i e^{it}\,dt = i\int_0^{\pi} (e^{it} + 1)\,dt$$

$$= i\left[\frac{e^{it}}{i} + t\right]_0^{\pi} = (e^{i\pi} + i\pi) - (e^{i0} + i0) = -1 + i\pi - 1$$

$$= -2 + \pi i$$