Answers for Practice Exam

1. a) see page 8: $w = 1 - i = \sqrt{2}\, e^{i5\pi/4}$

$$z = r e^{i\theta}$$

$$z^3 = r^3 e^{i3\theta} = \sqrt{2}\, e^{i5\pi/4}$$

$$r^3 = 2^{1/2} \qquad 3\theta = 5\pi/4 + 2k\pi$$

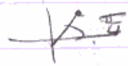
$$r = 2^{1/6} \qquad \theta = \frac{5\pi}{12} + \frac{2k\pi}{3} \qquad \theta_0 = \frac{5\pi}{12},\ \theta_1 = \frac{5\pi}{12} + \frac{2\pi}{3} = \frac{13\pi}{12}$$

$$\theta_2 = \frac{5\pi}{12} + \frac{4\pi}{3} = \frac{21\pi}{12}$$

$$z_0 = 2^{1/6}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

$$z_1 = 2^{1/6}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$$

$$z_2 = 2^{1/6}\left(\cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12}\right)$$



b) see page 21 $\log(1+i) = \log|1+i| + i\arg(1+i)$

$$= \log\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)$$

c) see page 29

$$\cos(1+i) = \frac{e^{i(1+i)} + e^{-i(1+i)}}{2} = \frac{e^{i-1} + e^{-i+1}}{2}$$

$$= \frac{e^{-1}e^{i} + e^{1}e^{-i}}{2} = \frac{e^{-1}(\cos 1 + i\sin 1) + e(\underbrace{\cos(-1)}_{=\cos 1} + i\underbrace{\sin(-1)}_{=-\sin 1})}{2}$$

$$= \frac{(e^{-1}+e)\cos 1}{2} + i\frac{(e^{-1}-e)}{2}\sin 1$$

d) $\operatorname{Im} e^{2+i} = \operatorname{Im} e^2(\cos 1 + i\sin 1) = \operatorname{Im}(e^2\cos 1 + i e^2 \sin 1)$

$$= e^2 \sin 1$$

2. $f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i2xy$, $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

Easy solution: use parametric equations: $x = t$, $y = t+1$.

Then image curve is given by

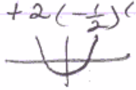
$$u = t^2 - (t+1)^2 = -2t - 1$$

$$v = 2t(t+1) = 2t^2 + 2t$$

Get rid of parameter t: $2t = -(u+1)$ $t = (-\tfrac{1}{2})(u+1)$

$\Rightarrow v = 2\cdot\frac{1}{4}(u+1)^2 + 2(-\frac{1}{2})(u+1) = \frac{1}{2}u^2 + u + \frac{1}{2} - u - 1$

$\Rightarrow v = \frac{1}{2}u^2 - 1$



3. $\cos z \cos w = \dfrac{e^{iz}+e^{-iz}}{2} \; \dfrac{e^{iw}+e^{-iw}}{2}$

$$= \frac{e^{i(z+w)} + e^{-i(z+w)} + e^{i(z-w)} + e^{-i(z-w)}}{4}$$

$\sin z \sin w = \dfrac{e^{iz}-e^{-iz}}{2i} \; \dfrac{e^{iw}-e^{-iw}}{2i}$

$$= -\frac{1}{4}\left(e^{i(z+w)} + e^{-i(z+w)} - e^{i(z-w)} - e^{-i(z-w)}\right)$$

$\cos z \cos w - \sin z \sin w = \frac{1}{2}\left(e^{i(z+w)} + e^{-i(z+w)}\right) + 0$

$= \cos(z+w)$

4. $f(z) = e^{3z} = e^{3(x+iy)} = e^{3x}(\cos 3y + i \sin 3y)$

$u = e^{3x}\cos 3y \qquad \frac{\partial u}{\partial x} = 3e^{3x}\cos 3y \qquad \frac{\partial u}{\partial y} = -e^{3x}(3\sin 3y)$

$v = e^{3x}\sin 3y \qquad \frac{\partial v}{\partial x} = 3e^{3x}\sin 3y \qquad \frac{\partial v}{\partial y} = e^{3x}(3\cos 3y)$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3e^{3x}\cos 3y + i\,3e^{3x}\sin 3y$

$= 3e^{3x}e^{i3y} = 3e^{3(x+iy)} = 3e^{3z}$.

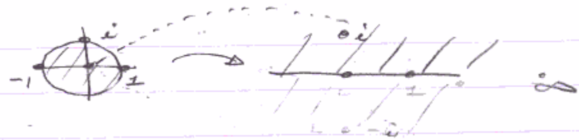
5. $u = x^2 - y^2 + 2x$

$\left.\begin{array}{ll} \frac{\partial u}{\partial x} = 2x+2 & \frac{\partial u}{\partial y} = -2y \\ \frac{\partial^2 u}{\partial x^2} = 2 & \frac{\partial^2 u}{\partial y^2} = -2 \end{array}\right\} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \Rightarrow v = 2yx + C(y)$

$\left.\begin{array}{l} \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x+2 \\ \hookrightarrow = 2x + C'(y) \end{array}\right\} \Rightarrow 2 = C'(y) \Rightarrow C(y) = 2y + D$

$v = 2yx + 2y + D$.

6. 

$(-1, i, 1) \longmapsto (0, 1, \infty)$

$$f(z) = \frac{z-(-1)}{z-1} \; \frac{i-1}{i-(-1)} = \frac{z+1}{z-1} \cdot \frac{i-1}{i+1} = \frac{z+1}{z-1} \cdot \frac{(i-1)^2}{2} = (-i)\frac{(z+1)}{(z-1)}$$

$f(0) = (-i) \cdot \frac{1}{(-1)} = i$ inside of circle $\rightarrow$ upper $\frac{1}{2}$-plane

7. $f(z) = |z| + i \operatorname{Arg} z = \sqrt{x^2+y^2} + i \operatorname{Tan}^{-1} \frac{y}{x}$

$u(x,y) = \sqrt{x^2+y^2}$

$v(x,y) = \operatorname{Tan}^{-1} \frac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{\frac{1}{2}(2x)}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right)^2 = \frac{1}{x^2+y^2}$$

$\Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ (e.g. let $x=0, y=1$ to see they are different)

Thus at least one of C-R equations doesn't hold $\Rightarrow$ f is not analytic.