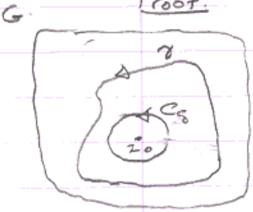
# Proof of Cauchy Formula

Given: $f$ analytic on region $G$ which has no holes

$\gamma$ closed curve, interior of $\gamma$ in $G$, which goes around $z_0$ once in positive direction

Theorem (Cauchy Formula)

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta$$



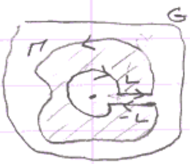

Proof: Let $F(\zeta) = \dfrac{f(\zeta)-f(z_0)}{\zeta - z_0}$. This is analytic, on $G \setminus \{z_0\}$. The curve

$$\Gamma = \gamma + L - C_\delta - L$$

doesn't go around $z_0$ hence

$$0 = \int_\Gamma F(\zeta)\,dz = \left(\int_\gamma + \int_L - \int_{C_\delta} - \int_L\right) F(\zeta)\,d\zeta$$

$$\Rightarrow \int_\gamma F(\zeta)\,dz = \int_{C_\delta} F(\zeta)\,dz.$$




$\Gamma = \gamma + L - C_\delta - L$

We have $\lim_{\zeta \to z_0} F(\zeta) = \lim_{\zeta \to z_0} \dfrac{f(\zeta) - f(z_0)}{\zeta - z_0} = f'(z_0)$

Choose $\delta_0 > 0$ so that $|\zeta - z_0| \le \delta_0 \Rightarrow |F(\zeta) - F(z_0)| \le 1$.

Then $|F(\zeta)| - |f'(z_0)| \le |F(\zeta) - f'(z_0)| \le 1$

$\Rightarrow |F(\zeta)| \le 1 + |f'(z_0)|$

Thus if $\delta \le \delta_0$, then

$$\left|\int_\gamma F(\zeta)\,d\zeta\right| = \left|\int_{C_\delta} F(\zeta)\,d\zeta\right| \le (1 + |f'(z_0)|)\,\overbrace{2\pi\delta}^{L = \text{circumference of } C_\delta}$$

Let $\delta \to 0$ and conclude $\int_\gamma F(\zeta)\,d\zeta = 0$. Thus

$$\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\,d\zeta = \int_\gamma \frac{f(z_0)}{\zeta - z_0} \overset{\oplus}{=} \int_{C_\delta} \frac{f(z_0)}{\zeta - z_0}\,d\zeta$$

where $\oplus$ follows by the $\Gamma$ discussion again. Finally

$$\int_{C_\delta} \frac{d\zeta}{\zeta - z_0} = \int_0^{2\pi} \frac{\delta i e^{i\theta}}{(z_0 + \delta e^{i\theta}) - z_0}\,d\zeta = 2\pi i$$