Additional Material you should know for final exam

Say f is a bounded function on $[a,b]$

Given partition P $a=x_0<\dots<x_n=b$, we define

$$L_P(f)=\sum_{i=1}^{n} m_i(x_i-x_{i-1}) \quad m_i=\inf\{f(x): x_{i-1}\le x\le x_i\}$$

$$U_P(f)=\sum_{i=1}^{n} M_i(x_i-x_{i-1}) \quad M_i=\sup\{f(x): x_{i-1}\le x\le x_i\}.$$

Since $m_i\le M_i$, we see that $L_P(f)\le U_P(f)$

Write $P\le Q$ if Q contains the points of P.

Lemma: If $P\le Q$, then $L_P(f)\le L_Q(f)\le U_Q(f)\le U_P(f)$

Proof: It suffices to consider the case Q has one more point: call it $\bar{x}$.

We may assume that

$$P:\ a=x_0<x_1<\dots<x_{i-1}<x_i<\dots<x_n$$

$$Q:\ a=x_0<x_1<\dots<x_{i-1}<\bar{x}<x_i<\dots<x_n$$

$$L_Q(f)=m_1(x_1-x_0)+\dots+m_i'(\bar{x}-x_{i-1})+m_i''(x_i-\bar{x})+m_{i+1}(x_{i+1}-x_i)+\dots$$

where $m_i'=\inf\{f(x): x_{i-1}\le x\le \bar{x}\}$

$m_i''=\inf\{f(x): \bar{x}\le x\le x_i\}$

We have $m_i'\ge m_i$ because m_i is the infimum over a larger set i.e, $[x_{i-1},x_i]\supseteq[x_{i-1},\bar{x}]$, and similarly $m_i''\ge m_i$. Thus

$$L_Q(f)\ge m_1(x_1-x_0)+\dots+m_i(\bar{x}-x_{i-1})+m_i(x_i-\bar{x})+m_{i+1}(x_{i+1}-x_i)+\dots$$

$$=m_1(x_1-x_0)+\dots+m_i(x_i-x_{i-1})+\dots=L_P(f).$$

You give reason for $U_Q(f)\le U_P(f)$. QED.

Cor: If P and Q are any partitions of $[a,b]$, then $L_P(f)\le U_Q(f)$.

Pf: Let R be the partition that contains the points of P and Q. Then

$$L_P(f)\le L_R(f)\le U_R(f)\le U_Q(f)$$

Def: f is Riemann integrable if

$$\sup_P L_P(f)=\inf_Q U_Q(f)$$

and we then let $\int_a^b f(x)\,dx=\sup_P L_P(f)=\inf_Q U_Q(f)$.

②

Theorem: If f is continuous on $[a,b]$, then it is Riemann integrable

Proof: It suffices to show there is a partition P such that

$$|U_P(f) - L_P(f)| \le \varepsilon.$$

f continuous on $[a,b] \Rightarrow f$ is uniformly continuous on $[a,b]$.

Let $\delta > 0$ be such that $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon / b-a$.

Let $P: a = x_0 < x_1 < \ldots < x_n = b$ be such that $|x_i - x_{i-1}| < \delta$

for all i. Since f is cont on $[x_{i-1}, x_i]$ we can

choose $x', x'' \in [x_{i-1}, x_i]$ with $f(x_i') = m_i$, $f(x_i'') = M_i$.

We have that $|x_i' - x_i''| < \delta \Rightarrow |f(x_i') - f(x_i'')| < \varepsilon / b-a$ and

$$U_P(f) - L_P(f) = \sum (M_i - m_i)(x_i - x_{i-1}) \le \frac{\varepsilon}{b-a} \sum (x_i - x_{i-1}) = \varepsilon.$$

QED.

Note that if x_i^* is an arbitrary point in $[x_{i-1}, x_i]$ then

$$m_i \le f(x_i^*) \le M_i$$

hence $\quad L_P(f) = \sum m_i (x_i - x_{i-1}) \le \sum f(x_i^*)(x_i - x_{i-1}) \le \sum M_i (x_i - x_{i-1}) = U_P$

We call a sum of the form

$$S_P = \sum f(x_i^*)(x_i - x_{i-1})$$

a Riemann sum. We have

$$L_P(f) \le \left\{ \begin{matrix} \int_a^b f(x)\,dx \\ S_P \end{matrix} \right\} \le U_P(f)$$

hence if $|U_P(f) - L_P(f)| < \varepsilon$, we have

$$|S_P - \int_a^b f(x)\,dx| < \varepsilon.$$

We can get a sequence $S_{P_k} \longrightarrow \int_a^b f(x)\,dx$ by choosing a

sequence P_k with $|U_{P_k} - L_{P_k}| \longrightarrow 0$. If f is continuous it suffices to

choose $P_k: a = x_0^{(k)} < x_1^{(k)} < \ldots < x_{n_k}^{(k)} = b$ with $\max |x_i^{(k)} - x_{i-1}^{(k)}| \longrightarrow 0$

Props: Say that f and g are cont. Then

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$$

$$f \ge 0 \Rightarrow \int_a^b f(x)\,dx \qquad [\text{Cor: } f \le g \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

$$a < c < b \Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof: Choose P_k as indicated above (see ($\Rightarrow$)), and let $x_i^{(k)*} \in [x_{i-1}^{(k)}, x_i^{(k)}]$

Then $S_{P_k}(f) \to \int_a^b f(x)\,dx$ $\quad S_{P_k}(g) \to \int_a^b g(x)$ $\quad S_{P_k}(f+g) \to \int_a^b f(x)+g(x)dx$

We have

$$\left.\begin{aligned} S_{P_k}(f) &= \sum_i f(x_i^{(k)*})(x_i^{(k)} - x_{i-1}^{(k)}) \\ S_{P_k}(g) &= \sum_i g(x_i^{(k)*})(x_i^{(k)} - x_{i-1}^{(k)}) \\ S_{P_k}(f+g) &= \sum \left(f(x_i^{(k)}) + g(x_i^{(k)})\right)(x_i^{(k)} - x_{i-1}^{(k)}) \end{aligned}\right\} \Rightarrow S_{P_k}(f+g) = S_{P_k}(f) + S_{P_k}(g)$$

so $\lim S_{P_k}(f+g) = \lim S_{P_k}(f) + \lim S_{P_k}(g) \Rightarrow \int_a^b (f+g)(x)\,dx = \int_a^b f(x) + \int_a^b g(x)\,dx$ $\blacksquare$

YOU PROVE OTHER ASSERTIONS

Integral Mean Value Theorem: f cont on $[a,b] \Rightarrow \exists\, c \in [a,b]$:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c).$$

Def: If f is defined on $[a,b]$ and $c \in (a,b)$ we say f is differentiable at c if $L = \lim\limits_{\substack{x\to c\\ x\neq c}} \dfrac{f(x)-f(c)}{x-c}$ exists. We then set $f'(c) = L$.

Thm: If f is differentiable at c, then it is continuous at c.

Pf: $f(x) = \dfrac{f(x)-f(c)}{x-c}(x-c) + f(c)$ for all $x \neq c$. Thus

$$\Rightarrow \lim_{\substack{x\to c\\ x\neq c}} f(x) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\lim_{x\to c}(x-c) + \lim_{x\to c} f(c) = f'(c)\cdot 0 + f(c) = f(c)$$

Thm: If f and g are diff at c, then so are $f+g$ and fg.

Pf: $$\frac{f(x)g(x) - f(c)g(c)}{x-c} = \frac{f(x)-f(c)}{x-c}\cdot g(x) + f(c)\,\frac{g(x)-g(c)}{x-c} \quad (x\neq c)$$

$$\lim_{\substack{x\to c\\ x\neq c}} \frac{f(x)g(x) - f(c)g(c)}{x-c} = f'(c)\lim_{\substack{x\to c\\ x\neq c}} g(x) + f(c)\,g'(x)$$

$$= f'(c)\,g(c) + f(c)g'(c) \quad \left[\begin{array}{l}\text{SINCE } g \text{ is}\\ \text{cont at } c\end{array}\right.$$

Thm: If g is diff at c and f is diff at $g(c)$, then $f\circ g$ is diff at c

Pf: $$\left[\frac{f(g(x)) - f(g(c))}{x-c} = \frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\;\frac{g(x)-g(c)}{x-c}\right]$$

This has to be modified since one might have $g(x) - g(c) = 0$ for $x \neq c$.

Let $F(y) = \begin{cases} \dfrac{f(y)-f(d)}{y-d} & \text{where } d = g(c), \text{ for } y \neq d \\ f'(d) & \text{for } y = d \end{cases}$

This is continuous at d, and

$$f(y) - f(d) = F(y)(y-d) \quad \text{is valid for } \underline{\underline{\text{all}}}\ y.$$

We have $d = g(c) \Rightarrow$

$$f(g(x)) - f(g(c)) = F(g(x))(g(x) - g(c))$$

Hence

$$\lim_{\substack{x \to c \\ x \neq c}} \frac{f(g(x)) - f(g(c))}{x-c} = \lim_{\substack{x \to c \\ x \neq c}} F(g(x)) \lim_{\substack{x \to c \\ x \neq c}} \frac{g(x)-g(c)}{x-c}$$

$$= \underbrace{F(g(c))}_{\substack{\text{because } F \text{ is cont at } d = g(c) \\ \text{and } g \text{ is cont at } c}} g'(c) = f'(g(c))\, g'(c)$$

QED

Thm: If f is diff at $c \in (a,b)$ and $f(c) = m$ or $f(c) = M$ (where $m = \min\{f(x) : x \in [a,b]\}$, $M = \max\{f(x) : x \in [a,b]\}$) then $f'(c) = 0$.

Pf: $f(c) = m \Rightarrow \dfrac{f(x)-f(c)}{x-c} = \dfrac{f(x)-m}{x-c}$

If $x > c$, $f(x) - m \geq 0$ & $x - c > 0 \Rightarrow \dfrac{f(x)-m}{x-c} \geq 0 \Rightarrow$

$$\lim_{\substack{x \to c \\ x > c}} \frac{f(x)-f(c)}{x-c} \geq 0$$

If $x < c$, $f(x) - m \geq 0$ & $x - c < 0 \Rightarrow \dfrac{f(x)-m}{x-c} \leq 0 \Rightarrow$

$$\lim_{\substack{x \to c \\ x < c}} \frac{f(x)-f(c)}{x-c} \leq 0$$

Then $f'(c) = \displaystyle\lim_{\substack{x \to c \\ x \neq c}} \frac{f(x)-f(c)}{x-c} = 0$. QED

Rolle's Thm: f cont on $[a,b]$ and f diff on (a,b), and $f(a) = f(b) = 0$ imply that $\exists\, c \in (a,b) : f'(c) = 0$

Pf: Let m, M be as above. Since $f(a) = 0$, $m \leq 0 \leq M$.

Say $M > 0$. Let $f(c) = M$. Then $c \neq a, b \Rightarrow c \in (a,b) \Rightarrow f'(c) = 0$

Say $m < 0$. Let $f(c) = m$. Then $c \neq a, b \Rightarrow c \in (a,b) \Rightarrow f'(c) = 0$

Say $M = m = 0$. Then $f(x) = 0$ for all x. Let $c \in (a,b)$ be arbitrary. Clear that $f'(c) = \displaystyle\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = 0$.