Math 131 8/2.

Some things you should know for second exam.

<u>Completeness</u> Axiom (ours!) If $S \subseteq \mathbb{R}$, $\emptyset \neq S$, and S is bounded above, then it has a <u>least</u> <u>upper</u> <u>bound</u>, denoted $\sup S$.

<u>Th:</u> If $b_0 = \sup S$, then for all $\varepsilon > 0$, $(b_0 - \varepsilon, b_0] \cap S \neq \emptyset$

<u>Pf:</u> Since $S \leq b_0$, we have $S \cap (b_0 - \varepsilon, b_0] = \emptyset$ implies $S \leq b_0 - \varepsilon$ contradicting the fact that b_0 is the <u>least</u> upper bound.

<u>Def.</u> $x_n \to L$ if $\forall \varepsilon > 0, \exists N: n \geq N \Rightarrow |x_n - L| < \varepsilon$

<u>Th:</u> $x_n \to L \Rightarrow x_n$ is bounded, i.e., $\exists K: |x_n| \leq K$ for <u>all</u> n.

<u>Pf:</u> Say $n \geq N \Rightarrow |x_n - L| < 1$. Then

$$|x_n| - |L| \leq |x_n - L| < 1 \Rightarrow |x_n| < 1 + |L|$$

Let $K = \max\{|x_1|, \dots, |x_{N-1}|, 1+|L|\}$.

<u>Th:</u> $x_n \to L$, $x_n \neq 0$, $L \neq 0 \Rightarrow \exists c > 0: |x_n| \geq c$ for <u>all</u> n.

<u>Pf:</u> Say $n \geq N \Rightarrow |x_n - L| < |L|/2$. Then $n \geq N \Rightarrow$

$$|L| - |x_n| \leq |L - x_n| < |L|/2 \Rightarrow |L|/2 < |x_n|.$$

Let $c = \min\{|x_1|, \dots, |x_{N-1}|, |L|/2\}$.

<u>Th:</u> $x_n \to L$ $y_n \to M \Rightarrow x_n + y_n \to L + M$, $x_n y_n \to LM$, and if $y_n \neq 0$, and $M \neq 0$, $\frac{1}{y_n} \to \frac{1}{M}$.

<u>Pf:</u> $+$: look at your notes

$$|x_n y_n - LM| = |x_n y_n - L y_n + L y_n - LM|$$
$$\leq |x_n - L|\,|y_n| + |L|\,|y_n - M|$$

Choose K with $|y_n| \leq K$ all n:

$$\leq |x_n - L| K + |L|\,|y_n - M|.$$

Given $\varepsilon > 0$ Choose $N_1: n \geq N_1 \Rightarrow |x_n - L| < \frac{\varepsilon}{2K}$

Choose $N_2: n \geq N_2 \Rightarrow |y_n - M| < \frac{\varepsilon}{2|L|}$. Then let $N = \max\{N_1, N_2\}$

$$n \geq N \Rightarrow |x_n y_n - LM| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\left|\frac{1}{y_n} - \frac{1}{M}\right| = \left|\frac{M - y_n}{y_n M}\right| = \frac{|y_n - M|}{|y_n|\,|M|}$$

Choose $c > 0$ such that $|y_n| \geq c$ for all n.

$$\leq \frac{|y_n - M|}{c|M|}$$

Choose N such that $n \geq N \Rightarrow |y_n - M| < c|M|\varepsilon$

$$n \geq N \Rightarrow \left|\frac{1}{y_n} - \frac{1}{M}\right| < \varepsilon.$$

Th: If $b_0 = \sup S$, then $\exists x_n \in S$ with $x_n \to b_0$.

Pf: Choose $x_n \in S \cap (b_0 - \frac{1}{n}, b_0]$ (see first result above)

Then $|x_n - b_0| < \frac{1}{n} \Rightarrow x_n \to b_0$.

Th: If $x_1 \le x_2 \le \dots \le b$, then x_n converges. [Similarly $x_1 \ge x_2 \ge \dots \ge b$ $\Rightarrow x_n$ converges]

Pf: We claim $x_n \to b_0 = \sup S$ where $S = \{x_1, x_2, \dots\}$.

Given $\varepsilon > 0$, choose N with $x_N \in S \cap (b_0 - \varepsilon, b_0]$.

Then $n \ge N \Rightarrow b_0 - \varepsilon < x_N \le x_n \le b_0 \Rightarrow |x_n - b_0| < \varepsilon$.

(B) Th: If x_n is an arbitrary sequence, it has a monotone subsequence.

Pf: Let $S \subseteq \mathbb{N}$ be defined by

$S = \{n \in \mathbb{N} : x_n > x_{n+k}$ for all $k \ge 1\}$ "peaking points"

Case 1 If S is infinite let $S = \{n_1, n_2, \dots\}$. Then $n_1 < n_2 \Rightarrow x_{n_1} > x_{n_2}$, $n_2 < n_3 \Rightarrow x_{n_2} > x_{n_3}$ so we get $x_{n_1} > x_{n_2} > x_{n_3} > \dots$

Case 2 If S is finite let $N = \max S$. Then $n > N \Rightarrow$ n not a peaking point. Let $n_1 = N+1$. Then n_1 not peaking $\Rightarrow \exists n_2 > n_1 : x_{n_2} \le x_{n_1}$. n_2 not peaking $\Rightarrow \exists n_3$ etc.

We get $x_{n_1} \ge x_{n_2} \ge x_{n_3} \ge \dots$

Def: x_n is Cauchy if $\forall \varepsilon, \exists N : m, n \ge N \Rightarrow |x_m - x_n| < \varepsilon$.

Th: If x_n is convergent, then x_n is Cauchy

Pf: Say $x_n \to L$. GIVEN $\varepsilon > 0$, Choose N: $n \ge N \Rightarrow |x_n - L| < \varepsilon/2$.

Then $m, n \ge N \Rightarrow |x_m - x_n| \le |x_m - L| + |L - x_n| < \varepsilon$.

(C) Th: If x_n is Cauchy then x_n is bounded

Pf: Choose N: $m, n \ge N \Rightarrow |x_m - x_n| \le 1$. Then in particular letting $n = N$, $|x_m| - |x_N| \le |x_m - x_N| \le 1 \Rightarrow |x_m| \le 1 + |x_N|$.

Let $K = \max\{|x_1|, \dots, |x_{N-1}|, 1 + |x_N|\}$. Then $|x_n| \le K$ all n.

(D) Th: If x_n is Cauchy and $x_{n_k} \to L$, then $x_n \to L$.

Pf: Given $\varepsilon > 0$, choose N such that $m, n \ge N \Rightarrow |x_m - x_n| < \varepsilon/2$. Choose K such that $k \ge K \Rightarrow |x_{n_k} - L| < \varepsilon/2$. We may also assume that $n_K \ge N$.

Then $n \ge n_K \Rightarrow$

$$|x_n - L| \le |x_n - x_{n_K}| + |x_{n_K} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$(n \ge n_K \ge N)$

Theorem: If x_n is Cauchy then x_n converges.

Pf: From Ⓑ we may assume x_{n_k} is a monotone subsequence

From Ⓒ x_{n_k} is bounded, From Ⓐ x_{n_k} converges.

From Ⓓ x_n converges. QED.

Theorem (BZW) If x_n is a bounded sequence it has a convergent subsequence x_{n_k}.

Pf: From Ⓑ there is a monotone subsequence x_{n_k}.

By assumption x_{n_k} is bounded, so from Ⓐ x_{n_k} converges. QED.

Def: $\lim\limits_{\substack{x \to c \\ x \neq c}} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0 : |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Th: $\lim\limits_{\substack{x \to c \\ x \neq c}} f(x) = L \iff$ for any sequence $x_n \to c$ we have $f(x_n) \to$

Pf: $\Rightarrow$) Given $\varepsilon > 0$, choose $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Given $x_n \to c$, choose N such that $n \geq N \Rightarrow |x_n - c| < \delta$. Then $n \geq N \Rightarrow |f(x_n) - L| < \varepsilon$

$\Leftarrow$) Say $\lim\limits_{\substack{x \to c \\ x \neq c}} f(x) = L$ is FALSE. Then

$$\exists \varepsilon > 0 : \forall \delta > 0 \quad |x - c| < \delta \not\Rightarrow |f(x) - L| < \varepsilon$$

Fix such an $\varepsilon > 0$. Then for each n we can choose x_n such that $|x_n - c| < \frac{1}{n}$ but $|f(x_n) - L| \geq \varepsilon$

$x_n \to c$ but $f(x_n) \not\to L$.

Def: f is cont. at c if $\lim\limits_{\substack{x \to c \\ x \neq c}} f(x) = f(c)$. EQUIVALENTLY

$x_n \to c \Rightarrow f(x_n) \to f(c)$. We say f is cont on S if it is continuous at each $x \in S$.

Th1: f cont on $[a,b] \Rightarrow f$ bdd on $[a,b]$

Th2: f cont on $[a,b] \Rightarrow f$ assumes max on $[a,b]$

Th3: f cont on $[a,b]$, $f(a) < y < f(b) \Rightarrow \exists c : f(c) = y$.