Additional things you should know for exam 2.

Theorem: Say that f is continuous on $[a, b]$. Then it is bounded on $[a, b]$.

Proof: Say that f is not bounded. Then for each n there is an $x_n \in [a, b]$ with $|f(x_n)| \geq n$. Let $x_{n_k} \to c$ (BZW). Then $f(x_{n_k}) \to f(c)$ (continuity) $\Rightarrow f(x_{n_k})$ is bounded, contradiction.

Theorem: Say that f is continuous on $[a, b]$. Then it must assume its maximum value, i.e., $\exists c \in [a, b]: f(c) \geq f(x)$ for all x.

Proof. Since f is bounded, there is a b with $f([a, b]) \leq b$. Thus we may let $b_0 = \sup f([a, b])$. Let $f(x_n) \to b_0$ where $x_n \in [a, b]$. Let $x_{n_k} \to c$. Then $f(x_{n_k}) \to b_0$ and $f(x_{n_k}) \to f(c)$ so $b_0 = f(c)$.

Theorem: Say that f is continuous on $[a, b]$ and $f(a) < y < f(b)$. Then there is a $c \in [a, b]$ with $f(c) = y$.

Proof: Let $S = \{x : f(x) \leq y\}$. $S \neq \emptyset$ because $a \in S$, and $S \leq b$. Thus we may let $c = \sup S$. We have $\exists x_n \in S$ such that $x_n \to c$. Thus $f(x_n) \to f(c)$ and since $f(x_n) \leq y$, we have $f(c) \leq y$. If $f(c) < y$ let $\epsilon = y - f(x)$. Then chose $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$ $\Rightarrow f(x) < f(c) + \epsilon$. In particular this is the case for $x = c + \delta/2$, and we have

$$c + \delta/2 \in S \leq c$$

a contradiction.

Theorem: Say f is continuous on $[a, b]$. Then it is uniformly continuous on $[a, b]$.

Proof: If f is not uniformly continuous on $[a, b]$, we can find $\epsilon > 0$ and $x_n, x_n' \in [a, b]$ such that $|x_n - x_n'| \to 0$ but $|f(x_n) - f(x_n')| \geq \epsilon$. Let $x_{n_k} \to c$ [BZW]. Then

$$|x_{n_k}' - c| \leq |x_{n_k}' - x_{n_k}| + |x_{n_k} - c| \to 0$$

so $x_{n_k}' \to c$. Thus $f(x_{n_k}) \to f(c)$, $f(x_{n_k}') \to f(c) \Rightarrow |f(x_{n_k}) - f(x_{n_k}')| \to 0$ contradicting $|f(x_{n_k}) - f(x_{n_k}')| \geq \epsilon$.