More things for the final   We write $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$ (i.e., drop "$\rightarrow$")

Th: $\| \|_1$, $\| \|_2$, and $\| \|_\infty$ are norms on $\mathbb{R}^p$

Pf See assignment 7 solutions for $\| \|_1$ and $\| \|_\infty$. For $\| \|_2$ we use the dot product on $\mathbb{R}^p$:

Def: $x \cdot y = x_1 y_1 + \cdots + x_p y_p$ Note $\|x\|_2 = (x \cdot x)^{\frac{1}{2}}$ [NOTE: $x \cdot x \geq 0$ TRIVIALLY]

It is immediate that $(cx + dy) \cdot z = c(x \cdot z) + d(y \cdot z)$ and the same is true for the right variable.

Schwarz Lemma: If $x, y \in \mathbb{R}^p$ then $|x \cdot y| \leq \|x\|_2 \|y\|_2$

Proof: $\forall x, y \in \mathbb{R}^p$, $0 \leq (x-y) \cdot (x-y) = x \cdot x - 2x \cdot y + y \cdot y$

hence $x \cdot y \leq \dfrac{\|x\|^2 + \|y\|^2}{2}$. Also

$$-(x \cdot y) = (-x) \cdot y \leq \frac{\|x\|^2 + \|(-1)y\|^2}{2} = \frac{\|x\|^2 + \|y\|^2}{2}$$

hence $|x \cdot y| \leq \dfrac{\|x\|^2 + \|y\|^2}{2}$. It follows that for any $t > 0$

$$|x \cdot y| = t^{\frac{1}{2}} x \cdot t^{-\frac{1}{2}} y \leq \frac{\|t^{\frac{1}{2}} x\|^2 + \|t^{-\frac{1}{2}} y\|^2}{2}$$

$$= \frac{t\|x\|^2 + t^{-1}\|y\|^2}{2}$$

We may assume both $x \neq \vec{0} = (0, \ldots, 0)$ and $y \neq \vec{0}$ since otherwise desired inequality is just $0 \leq 0$. Let $t = \dfrac{\|y\|}{\|x\|}$ and we get

$$|x \cdot y| \leq \frac{\|x\| \|y\| + \|x\| \|y\|}{2}$$ QED. for Lemma

Proof of Theorem (continued)

"$\|(0, \ldots, 0)\|$"

1) $\|x\|_2 = 0 \iff \sqrt{\Sigma |x_i|^2} = 0 \iff x_i = 0$ for all $i \iff x = \vec{0}$

2) $\|cx\|_2 = \sqrt{(cx) \cdot (cx)} = \sqrt{c^2 (x \cdot x)} = c\sqrt{x \cdot x} = c\|x\|_2$

3) $\|x+y\|_2^2 = (x+y) \cdot (x+y)$

$$= x \cdot x + 2x \cdot y + y \cdot y$$

$$\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 = (\|x\|_2 + \|y\|_2)^2$$

– take square root –   QED.

Lemma: $x \in \mathbb{R}^p \Rightarrow \|x\|_1 \leq \sqrt{p} \|x\|_2$, $\|x\|_2 \leq \sqrt{p} \|x\|_\infty$, $\|x\|_\infty \leq \|x\|_1$.

Pf: $\|x\|_1 = \sum_{i=1}^{p} |x_i| = \sum_{i=1}^{p} |x_i| \cdot 1 = (|x_1|, \ldots, |x_p|) \cdot (1, \ldots, 1) \leq$

hence $\|x\|_1 = (|x_1|,\ldots,|x_p|)\cdot(1,\ldots,1)$

$$\le \|(|x_1|,\ldots,|x_p|)\|_2 \; \|(1,\ldots,1)\|_2 \quad \text{(Schwarz ineq)}$$

$$= (x_1^2+\cdots+x_p^2)^{\frac{1}{2}} \, (1^2+\cdots+1^2)^{\frac{1}{2}} \quad (|x_i|^2 = x_i^2)$$

$$= \|x\|_2 \sqrt{p}$$

$$\|x\|_2^2 = \sum_{i=1}^{p} |x_i|^2 \le p \max|x_i|^2 = p(\max|x_i|)^2 = p\|x\|_\infty^2$$

take square root

Say $\max|x_i| = |x_{i_0}|$. Then

$$\|x\|_\infty = |x_{i_0}| \le \sum |x_i| = \|x\|_1 .\quad \text{QED.}$$

Def If (M,d) is a metric space and $x_n \in M$, $c \in M$ we define

$\lim_{n\to\infty} x_n = c$ if $\forall \varepsilon>0, \exists N: n \ge N \Rightarrow d(x_n,c)<\varepsilon$ $\left[\Leftrightarrow d(x_n,c)\to 0\right]$

and then we say x_n converges (to c) and write $x_n \to c$.

We say x_n is Cauchy if $\forall \varepsilon>0, \exists N: m,n \ge N \Rightarrow d(x_m,x_n)<\varepsilon$.

Given metric spaces (M,d) and (N,ρ) and $c \in M$

We say $f: M \to N$ is continuous at c

$$\forall \varepsilon>0, \exists \delta>0: d(x,c)<\delta \Rightarrow \rho(f(x),f(c))<\varepsilon.$$

We say $f: M \to N$ is continuous on M if it is continuous at each $c \in M$, i.e.,

$$\forall c\in M, \forall \varepsilon>0, \exists \delta>0 \; \forall x \in M, \; d(x,c)<\varepsilon \Rightarrow \rho(f(x),f(c))<\varepsilon$$

We say $f: M \to N$ is uniformly continuous if

$$\forall \varepsilon>0, \exists \delta>0: \forall x,x' \in M, \; d(x,x')<\delta \Rightarrow \rho(f(x),f(x'))<\varepsilon.$$

Thm: x_n converges $\Rightarrow$ x_n is Cauchy [see proof for $\mathbb{R}$]

Def. A metric space (N,d) is complete if each Cauchy sequence in N converges. E.G. We have PROVED $\mathbb{R}$ is complete!

Def: A subset N of a metric space (M,d) is closed if $x_n \in N$ and $x_n \to x \in M \Rightarrow x \in N$.

Note change of notation

NOTE: In $\mathbb{R}^p$ we have $x_n = (x_n(1),\ldots,x_n(p))$, $x = (x(1),\ldots,x(p))$

$$\|x_n - x\|_1 \to 0 \Leftrightarrow \|x_n - x\|_2 \to 0 \Leftrightarrow \|x_n - x\|_\infty \to 0$$

$$\Leftrightarrow |x_n(i) - x(i)| \to 0 \quad (i=1,\ldots p)$$

hence convergence in one metric implies all metrics and $x_n \to x \Leftrightarrow x_n(i) \to x(i)$ for $i=1,\ldots,p$.

Applying same to $x_n - x_m$ we see

x_n is Cauchy for $\| \ \|_1$, $\| \ \|_2$, or $\| \ \|_\infty$

$\iff x_n(i)$ is Cauchy $(i = 1, \dots, p)$

Th: $\mathbb{R}^p$ is complete.

Pf: Say $x_n \in \mathbb{R}^p$. Then

x_n Cauchy $\Rightarrow x_n(i) \in \mathbb{R}$ Cauchy $\Rightarrow x_n(i)$ converges (since $\mathbb{R}$ is complete). Let $x(i) = \lim x_n(i)$, and $x = (x(1), \dots, x(p))$. Then $x_n \to x$.

Th: Say (M, d) is a metric space and $N \subseteq M$ is given the restricted metric (which we also denote by d). Then

(1) If N is complete, it is closed

(2) If M is complete and $N \subseteq M$ is closed then N is complete.

Pf (1) Say $x_n \in N$ and $x_n \to x \in M$. Then latter implies x_n is Cauchy (in M and thus in N) hence N complete $\Rightarrow x_n \to y \in N$.

It follows that $y = x$ and thus $x \in N$.

(2) Say x_n is Cauchy in N. Then it is Cauchy in M and $x_n \to x \in M$. Since N is closed $x_n \in N$. QED.

Co: If $N \subseteq \mathbb{R}^p$, then N is a complete metric space $\iff N$ is closed in $\mathbb{R}^p$.

Th: We say a metric space M is compact if for any sequence $x_n \in M$ we have x_n has a convergent subsequence $x_{n_k} \to x \in M$.

We proved that $[a, b]$ is compact (BW theorem). $\mathbb{R} \begin{cases} \text{complete} \\ \text{NOT compact} \end{cases}$

Th: M compact $\Rightarrow M$ complete

Pf: Let $x_n \in M$ be Cauchy. Then $\exists$ subseq $x_{n_k} \to x \in M$. It follows that $x_n \to x$ (see our proof in $\mathbb{R}$).

Th: Say M is compact, and $f : M \to \mathbb{R}$ is continuous. Then (1) f is bounded (2) f assumes minimum and maximum values and (3) f is uniformly continuous.

These are proved in *exactly* the same manner as for $f: [a,b] \to \mathbb{R}$.

(1) If f is not bounded, choose $x_n \in \mathcal{M}$ with $|f(x_n)| \geq n$. Let $x_{n_k} \to x \in \mathcal{M}$ [$\mathcal{M}$ is compact]. Then $f(x_{n_k}) \to f(x)$ since f is continuous. Then $f(x_{n_k})$ is a bounded sequence [fact we showed in $\mathbb{R}$]. But $|f(x_{n_k})| \geq n_k$ contrad.

(2) Since $f(\mathcal{M})$ is bounded we can let

$$m = \inf f(\mathcal{M}) \qquad M = \sup f(\mathcal{M})$$

We have $\exists y_n \in f(\mathcal{M})$ with $y_n \to M$. Let $y_n = f(x_n)$ and let $x_{n_k} \to x$. Then $f(x_{n_k}) \to f(x)$. But $f(x_{n_k}) \to M \Rightarrow f(x) = M$. Same for m.

(3) Say f is *not* unif contt. Then

$\rho(f(x), f(x'))$

$$\exists \varepsilon > 0 : \forall \delta > 0,\ d(x,x') < \delta \not\Rightarrow |f(x) - f(x')| < \varepsilon.$$

Thus we can choose $x_n, x_n' \in \mathcal{M}$ with

$$d(x_n, x_n') < \frac{1}{n} \quad \text{but} \quad |f(x_n) - f(x_n')| \geq \varepsilon \text{ *for all n*}$$

Let $x_{n_k} \to x$, i.e., $d(x_{n_k}, x) \to 0$. Then

$$d(x_{n_k}', x) \leq d(x_{n_k}', x_{n_k}) + d(x_{n_k}, x)$$
$$< \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \to 0$$

so $x_{n_k}' \to x$. It follows that $f(x_{n_k}) \to f(x)$ and $f(x_{n_k}') \to f(x) \Rightarrow$

$$|f(x_{n_k}) - f(x_{n_k}')| \to 0 \quad \text{contradicting}$$

Cor: If (M,d) is compact, then it is *bounded* i.e., there is a point $x_0 \in M$ and a constant K such that $d(x, x_0) \leq K$ for all $x \in M$.

Pf: Let x_0 be *arbitrary*. In assignment 7 we proved that $x_n \to x \Rightarrow d(x_n, x_0) \to d(x, x_0)$ [this is a special case]. Then $f(x) = d(x, x_0)$ determines a continuous function $f: M \to \mathbb{R}$. From (2) above, f must be bounded.

(or complete)

Th: A subset $N \subseteq \mathbb{R}^k$ is compact $\Longleftrightarrow$ it is closed and bounded.

This is easy, but *we have run out of time*! Note that if S is infinite and $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$, S is complete and bounded but it is not compact.