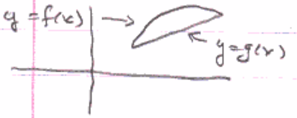
MORE THINGS YOU SHOULD KNOW FOR FINAL

Mean Value Theorem: Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Proof: Let $y = g(x)$ be straight line with slope $m = \frac{f(b)-f(a)}{b-a}$ and point $(a, f(a))$. We have



$$\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow g(x) = y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

We have $g'(x) = \frac{f(b)-f(a)}{b-a}$ and $g(a) = f(a)$, $g(b) = f(b)$.

Let $h(x) = f(x) - g(x)$. Then $h(a) = f(a) - g(a) = 0$, $h(b) = f(b) - g(b) = 0$

h is cont on $[a,b]$ and diff on (a,b) since this is obvious for $g(x)$.

Rolle's Thm $\Rightarrow \exists\, c \in (a,b): h'(c) = 0$, i.e.,

$$0 = f'(c) - g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}. \quad \text{QED}$$

Cor: If f is cont on $[a,b]$ and $f'(x) = 0$ on (a,b), then $f(x)$ is constant fn

Pf. Say $a < b_1 \le b$. From MVT, $\frac{f(b_1)-f(a)}{b_1-a} = f'(c)$ for some $c \in [a, b_1]$ and thus $f(b_1) = f(a)$.

Cor: If f is cont on $[a,b]$ and $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing, i.e., if $a \le a_1 < b_1 \le b$ then $f(a_1) < f(b_1)$.

Pf: Follows from MVT (you give details!)

Cor: If f, g cont on $[a,b]$ and $f'(x) = g'(x)$ for all $x \in (a,b)$, then there is a constant C such that $g(x) = f(x) + C$

Pf: You prove this. [Consider $h(x) = g(x) - f(x)$].

FUND Thm. of Calculus (I) Say F is differentiable on $[a,b]$ and $F'(x) = f(x)$, where f is continuous on $[a,b]$.

$$F(b) - F(a) = \int_a^b f(x)dx$$

Pf: Let $P: a = x_0 < x_1 < \dots < x_n = b$. MVT ($n$ times) $\Rightarrow \exists\, c_i \in [x_{i-1}, x_i]$:

$$F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = S_P(f)$$

where $S_P(f)$ is the Riemann sum determined by P and $c_i \in [x_{i-1}, x_i]$.
Let P_n be a partition with $\max |x_i^{(n)} - x_{i-1}^{(n)}| < \frac{1}{n}$.
[We have shown that $\int_a^b f(x)\,dx = \lim_{n\to\infty} S_{P_n}(f)$ Regardless of which intermediate points you use] so
$$F(b) - F(a) = S_{P_n}(f) \xrightarrow[n\to\infty]{} \int_a^b f(x)\,dx.$$

<u>Fund Thm of Calculus (II)</u> Say that f is cont on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$ for all $x \in [a,b]$.

~~Pf~~ Say that $c \in [a,b]$. Then

$\underline{x > c} \Rightarrow$ $$\frac{F(x) - F(c)}{x-c} = \frac{1}{x-c}\left[\int_a^x f(t)\,dt - \int_a^c f(t)\,dt\right]$$
$$= \frac{1}{x-c}\left[\int_c^x f(t)\,dt\right] = f(c^*) \qquad c^* \in [c,x]$$

(by integral MVT). Thus
$$\lim_{\substack{x\to c\\ x>c}} \frac{F(x)-F(c)}{x-c} = \lim_{\substack{x\to c\\ x>c}} f(c^*) = f(c) \qquad [f \text{ is cont}]$$

$\underline{x<c} \Rightarrow$ $$\frac{F(x) - F(c)}{x-c} = \frac{F(c)-F(x)}{c-x} = \frac{1}{c-x}\left[\int_a^c f(t)\,dt - \int_a^x f(t)\,dt\right]$$
$$= \frac{1}{c-x}\int_x^c f(t)\,dt = f(c^*) \qquad c^* \in [x,c]$$

$$\lim_{\substack{x\to c\\ x<c}} \frac{F(x)-F(c)}{x-c} = \lim_{\substack{x\to c\\ x<c}} f(c^*) = f(c). \qquad \text{QED.}$$

<u>Normed spaces / Metric spaces</u>

$\mathbb{R}^n$ $\vec{x} = (x_1, \ldots, x_n)$ $x_i \in \mathbb{R}$ $\vec{x}+\vec{y}$, $c\vec{x}$ defined in usual way

A <u>norm</u> on a vector space V is a map
$$\|\ \| : V \to [0,\infty)$$
such that N1 $\|x\| = 0 \Leftrightarrow x = 0$

N2 $\|x+y\| \le \|x\| + \|y\|$

N3 $\|cx\| = |c|\,\|x\|$.

3 norms on $\mathbb{R}^n$: Say $\vec{x} = (x_1, \ldots, x_n)$

$\|\vec{x}\|_1 = \sum |x_k|$

$\|\vec{x}\|_2 = \left(\sum |x_k|^2\right)^{\frac{1}{2}}$ (N2 is not trivial - see below)

$\|\vec{x}\|_\infty = \max |x_k|$