Printed Name__________ Signature__________ UID______

Math 131a/2 W02 Second Hour Exam

1(20). a) Define $\lim_{n\to\infty} x_n = c$

For all $\varepsilon>0$, $\exists N\in\mathbb{N}$: $n\geq N \Rightarrow |x_n-c|<\varepsilon$

b) Define: x_n is a Cauchy sequence

For all $\varepsilon>0$, $\exists N\in\mathbb{N}$: $m,n\geq N \Rightarrow |x_m-x_n|<\varepsilon$

c) Define: $\lim_{\substack{x\to c\\ x\neq c}} f(x) = L$

For all $\varepsilon>0$, $\exists\delta>0$: $|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$

d) Define: f is continuous at c.

$\lim_{\substack{x\to c\\ x\neq c}} f(x) = f(c)$

e) Prove that if $f(x_n)\to L$ for any sequence x_n such that $x_n\to c$, then $\lim_{\substack{x\to c\\ x\neq c}} f(x) = L$

Say it is false that $\lim_{\substack{x\to c\\ x\neq c}} f(x) = L$, i.e., $\sim[\forall\varepsilon>0, \exists\delta>0: |x-c|<\delta \Rightarrow |f(x)-c|<\varepsilon]$

Then $\exists\varepsilon>0$: for all $\delta>0$, $|x-c|<\delta \not\Rightarrow |f(x)-L|<\varepsilon$

Fix such an $\varepsilon>0$. For each $n\in\mathbb{N}$ choose x_n such that $|x_n-c|<\frac{1}{n}$ but $|f(x_n)-L|\geq\varepsilon$. Then $x_n\to c$ but $f(x_n)\not\to L$.

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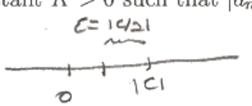
2(20).a) Show that if a_n is a sequence such that $a_n\neq 0$, and $a_n\to c\neq 0$, then there is a constant $K>0$ such that $|a_n|\geq K$ for all n.

We know $|a_n|\to|c|$



Choose N such that $n\geq N \Rightarrow ||a_n|-|c||<|c|/2$.

Then $|c|-|a_n|<|c|/2 \Rightarrow |a_n|>|c|-|c|/2 = |c|/2$

Let $K=\min\{|a_1|,\ldots,|a_{N-1}|,|c|/2\}$

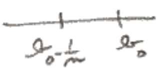
b) Prove that under the assumptions of a), $1/a_n \to 1/c$.

$$\left|\frac{1}{a_n}-\frac{1}{c}\right| = \left|\frac{c-a_n}{a_n c}\right| = \frac{|c-a_n|}{|a_n||c|} \le \frac{|c-a_n|}{K|c|}$$

Given $\varepsilon>0$, choose N so that $n \geq N \Rightarrow$
$|c-a_n| < \varepsilon |c| K$. Then

$$n \geq N \Rightarrow \left|\frac{1}{a_n}-\frac{1}{c}\right| < \frac{\varepsilon K|c|}{K|c|} = \varepsilon.$$

3(20). Show that if $S \neq \emptyset$ is a bounded set and $b_0 = \sup S$, then there is a sequence $s_n \in S$ such that $s_n \to b_0$.



For each $n \in \mathbb{N}$, $(b_0-\frac{1}{n}, b_0] \cap S \neq \emptyset$

[otherwise $b_0-\frac{1}{n} \geq S$, contradiction of sup]

Choose $s_n \in S \cap (b_0-\frac{1}{n}, b_0] \cap S$. Then

$|s_n - b_0| < \frac{1}{n} \Rightarrow s_n \to b_0$.

4(20). Prove the Balzano-Weierstrass theorem, i.e., prove that if $a_n \in \mathbb{R}$ is a bounded sequence, then it must have a convergent subsequence.

a_n has a monotone subsequence, a_{n_k}.

Since it is bounded, it converges.

5(20) a) Prove that if f is a continous function on $[a, b]$, then it must be bounded.

Say f is not bounded. Choose x_n with $|f(x_n)| \geq n$.

Let $x_{n_k} \to c$ (Bolzano–W). Then $f(x_{n_k}) \to f(c)$

[f is continuous] $\Rightarrow f(x_{n_k})$ bounded.

But $|f(x_{n_k})| > n_k$ contrad.

b) Is it necessary to assume that f is continuous in this theorem? Prove your assertion.

Yes — here is a counter example to the theorem without continuity:

$$f(x) = \begin{cases} \frac{1}{x} & 0 < x < 1 \\ 0 & x = 0 \end{cases}$$