Printed Name____________________Signature____________________UID______

Math 131a/2 W02 First Hour Exam Feb. 4, 2002

1(10). Prove that for any sets $A, B \subseteq X$,

$$X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B).$$

[Hint: Begin $x \in X \backslash (A \cup B) \Leftrightarrow \ldots$]

$(\forall x \in X)$

$x \in X \backslash (A \cup B) \Leftrightarrow \sim (x \in A \text{ or } x \in B)$

$\Leftrightarrow \sim (x \in A)$ and $\sim (x \in B)$

$\Leftrightarrow x \in X \backslash A$ and $x \in X \backslash B$

$\Leftrightarrow x \in (X \backslash A) \cap (X \backslash B)$

2(30). Define:

a) $f : X \to Y$ is a one-to-one function

$(\forall x_1, x_2 \in X)\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

b) $f : X \to Y$ is an onto function

$(\forall y \in Y)(\exists x \in X) f(x) = y$

c) $f : X \to Y$ is a one-to-one correspondence

f is one-to-one and onto

d) X and Y have the same cardinality $(X \approx Y)$

there exists a one-to-one correspondence $f : X \to Y$

e) X is a finite set

$X = \emptyset$ or X has the same cardinality as $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$

f) X is countable.

$X \approx \mathbb{N}$

2(10). Prove that for any real numbers x and y, $||x| - |y|| \leq |x - y|$.

$(|x| = |x - y + y| \leq |x - y| + |y|) \Rightarrow |x| - |y| \leq |x - y|$

Interchanging x and y, $|y| - |x| \leq |y - x| = |x - y|$

$\Downarrow$ ⊕

$||x| - |y|| \leq |x - y|$

⊕ use $|a| \leq c \Leftrightarrow a \leq c$ and $-a \leq c$

$(\Leftrightarrow -c \leq a \leq c)$

3(20). Prove that if $f : \mathbb{N} \to Y$ is an onto function, and Y is infinite, then Y is countable.

Define $g : Y \to \mathbb{N}$ by letting $g(y)$ be an $n \in \mathbb{N}$ such that $f(n) = y$. Then $f(g(y)) = y$. It follows g is 1-1 since $g(y_1) = g(y_2) \Rightarrow f(g(y_1)) = f(g(y_2)) \Rightarrow y_1 = y_2$.

Let $S = g(Y)$. Then $g : Y \approx S$. But $S \subseteq \mathbb{N}$ infinite $\Rightarrow S \approx \mathbb{N}$ so $Y \approx \mathbb{N}$.

4(20). Prove that $\mathbb{N} \times \mathbb{N}$ is countable.

Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(m, n) = 2^m 3^n$.

Then f is 1-1 since

$$f(m_1, n_1) = f(m_2, n_2) \Rightarrow 2^{m_1} 3^{n_1} = 2^{m_2} 3^{n_2} \xRightarrow{\text{FTA}} m_1 = m_2 \text{ and } n_1 = n_2$$

Thus if $S = f(\mathbb{N} \times \mathbb{N})$ then $\mathbb{N} \times \mathbb{N} \approx S$. Since $S \subseteq \mathbb{N}$ is infinite, $S \approx \mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.

5(10). Prove that 24 does not have an ~~integer~~ rational cube root.

$$\sqrt[3]{24} = \frac{m}{n} \Rightarrow m^3 = 24 n^3$$

Let $m = 2^{a_1} 3^{a_2} 5^{a_3} \ldots \qquad 24 = 2^3 3$

$n = 2^{b_1} 3^{b_2} 5^{b_3} \ldots$

Then $2^{3a_1} 3^{3a_2} 5^{3a_3} \ldots = 2^3 3 (2^{3b_1} 3^{3b_2} \ldots)$

$= 2^{3+3b_1} 3^{1+3b_2} \ldots$

$\xRightarrow{\text{FTA}} 3a_2 = 1 + 3b_2 \Rightarrow a_2 = b_2 + \frac{1}{3}$ contradiction since $a_2, b_2 \in \mathbb{N}$.