Assignment 7

p.202 1. We use the notation d_1 and d_∞. We have for $x=(x_1,x_2)$ $y=(y_1,y_2)$

Def $d_1(x,y) = \|x-y\|_1 = \sum_{i=1}^{m} |x_i-y_i| = |x_1-y_1|+|x_2-y_2|$

$d_\infty(x,y) = \|x-y\|_\infty = \max\{|x_1-y_1|, |x_2-y_2|\}$

In general given a normed space $(V, \|\ \|)$ we have seen that

$d(x,y) = \|x-y\|$

is a metric on V. Thus we have to show that $\|\ \|_1$ and $\|\ \|_\infty$ are norms.

a) $\|(x_1,x_2)\|_1 = 0 \Leftrightarrow |x_1|+|x_2| = 0 \Leftrightarrow x_1 = x_2 = 0 \Leftrightarrow (x_1,x_2)=(0,0)$

$\|c(x_1,x_2)\|_1 = \|(cx_1,cx_2)\|_1 = |cx_1|+|cx_2|$

$= |c|\,\|x_1\| + |c|\,\|x_2\|$

$\|(x_1,x_2)+(y_1,y_2)\|_1 = \|(x_1+x_2, y_1+y_2)\|_1$

$= |x_1+x_2| + |y_1+y_2|$

$\leq |x_1|+|x_2|+|y_1|+|y_2|$

$= \|(x_1,y_1)\|_1 + \|(x_2,y_2)\|_1$

b) $\|(x_1,x_2)\|_\infty = 0 \Leftrightarrow \max\{|x_1|,|x_2|\} = 0 \Leftrightarrow x_1=x_2=0 \Leftrightarrow (x_1,x_2)=(0,0)$

$\|c(x_1,x_2)\|_\infty = \max\{|cx_1|, |cx_2|\} = |c|\max\{|x_1|,|x_2|\} = |c|\,\|(x_1,x_2)\|_\infty$

$\|(x_1,x_2)+(y_1,y_2)\|_\infty = \|(x_1+y_1, x_2+y_2)\|_\infty$

$= \max\{|x_1+y_1|, |x_2+y_2|\}$

$\leq \max\{|x_1|+|y_1|, |x_2|+|y_2|\}$

$\leq \max\{|x_1|,|x_2|\} + \max\{|y_1|,|y_2|\}$

$= \|(x_1,x_2)\|_\infty + \|(y_1,y_2)\|_\infty$

The last inequality follows from

$|x_1|+|y_1| \leq \max\{|x_1|,|x_2|\} + \max\{|y_1|,|y_2|\}$

and $|x_2|+|y_2| \leq \max\{|x_1|,|x_2|\} + \max\{|y_1|,|y_2|\}$.

3. We define a new norm $\|\ \|_{L^1}$ on $C([a,b])$ by

$\|f\|_{L^1} = \int_a^b |f(x)|\,dx$

Since $\rho(f,g) = \|f-g\|_{L^1}$ it suffices to show $\|\ \|_{L^1}$ is a norm.

$\|f\|_{L^1}=0 \Rightarrow \int_a^b |f(x)|\,dx \Rightarrow f=0$ since f is continuous (see earlier assignment)

$\|cf\|_{L^1} = \int_a^b |cf(x)|\,dx = \int_a^b |c|\,|f(x)|\,dx = |c|\int_a^b |f(x)|\,dx$

$= |c|\,\|f\|_{L^1}$.

$\|f+g\|_{L^1} = \int_a^b |f(x)+g(x)|\,dx \leq \int_a^b |f(x)|+|g(x)|\,dx = \int_a^b |f(x)|\,dx + \int_a^b |g(x)|\,dx$

7. Say that $x_n \to x$ and $x_n \to y$. Given $\varepsilon > 0$ choose N so that $n \geq N \Rightarrow d(x_n, x) < \varepsilon/2$ and $d(x_n, y) < \varepsilon/2$.
Then $d(x,y) \leq d(x, x_N) + d(x_N, y) < \varepsilon$ (we don't use other n)
and since $\varepsilon > 0$ is arbitrary, $d(x,y) = 0$, hence $x = y$ (see def of metric space).

IN remaining problems we'll follow Reed: let "ρ" stand for a metric

8. $|\rho(x_n, y_n) - \rho(x,y)| \leq |\rho(x_n, y_n) - \rho(x, y_n)| + |\rho(x, y_n) - \rho(x,y)|$

we have the general fact that

$$|\rho(x,z) - \rho(y,z)| \leq \rho(x,y)$$

because

$$\rho(x,z) - \rho(y,z) \leq \rho(x,y)$$

Since $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ (Δ inequality),
and interchanging y and x,

$$\rho(y,z) - \rho(x,z) \leq \rho(y,x) = \rho(x,y)$$

Thus

$$|\rho(x_n, y_n) - \rho(x,y)| \leq |\rho(x_n, x) + \rho(y_n, y)$$

Choose N so that $n \geq N \Rightarrow \rho(x_n, x) < \varepsilon/2$ and $\rho(y_n, y) < \varepsilon/2$.
Then $n \geq N \Rightarrow |\rho(x_n, y_n) - \rho(x,y)| < \varepsilon$.

9. AS STATED THIS CLAIM IS FALSE. CONSIDER THE SEQUENCE $x_n = n$ in the metric space $\mathbb{N}$ (usual metric $\rho(m,n) = |m-n|$).
ACCORDING TO THE BOOKS INCORRECT DEFINITION, WE HAVE 1 IS A LIMIT POINT OF x_n
BUT CLEARLY $|x_{n_k} - 1| = |n_k - 1| \to \infty$.

10 a) we showed that δ is a metric in class
If $\delta(x,y) < 1$, then $x = y$.
b) In $\mathbb{R} = \mathbb{R}^1$, $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|_\infty = |\cdot|$ (usual absolute value) and $\rho_1 = \rho_2 = \rho_\infty = \rho$ (usual distance)
we have for any x, $\rho(x, x + \frac{1}{n}) < \frac{1}{n}$

c) 1–4 are not discrete (obvious)

5: DISCRETE SINCE $\rho(x,y) \in \mathbb{N} \cup \{0\} \Rightarrow$ if $\rho(x,y) < 1$ then $\rho(x,y) = 0$ hence $x = y$.

d) Say $x_n \to c$. Since (M,ρ) is discrete we can choose ε so that $\rho(x,c) < \varepsilon \Rightarrow x = c$. Choose N so that $n \geq N \Rightarrow \rho(x_n, c) < \varepsilon$. Then $n \geq N \Rightarrow x_n = c$. Thus the only convergent sequences are those which are eventually constant.

p. 209: 1 We proved this in lecture

2. We use the fact that $|\rho(x,z) - \rho(y,z)| \leq \rho(x,y)$ (see above). We have

$$|\rho(x_m, y_m) - \rho(x_n, y_n)| \leq |\rho(x_m, y_m) - \rho(x_n, y_m)| + |\rho(x_n, y_m) - \rho(x_n, y_n)|$$
$$\leq \rho(x_m, x_n) + \rho(y_m, y_n)$$

Choose N so that $m, n \geq N \Rightarrow \rho(x_m, x_n) < \varepsilon/2$ and $\rho(y_m, y_n) < \varepsilon/2$. Then

$$m, n \geq N \Rightarrow |\rho(x_m, y_m) - \rho(x_n, y_n)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

i.e., $\rho(x_n, y_n)$ is a Cauchy sequence of reals. Since $\mathbb{R}$ is complete, $\rho(x_n, y_n)$ converges.

3. Since $\mathbb{R}$ is complete it suffices to check whether the set is closed in $\mathbb{R}$. JUST $(0,\infty)$ and $\mathbb{Q}$ are not closed.

4. Since $\mathbb{R}^2$ is complete it suffices to check which sets are closed. (a) is not closed since $(1-\frac{1}{n}, 0)$ is in the set and $(1-\frac{1}{n}, 0) \to (1,0)$ which is not in the set. (b)–(d) are closed. E.G.: Say $f(x_n, y_n) = 0$. Then let $(x_n, y_n) \to (x,y) \in \mathbb{R}^2$. We have $f(x_n, y_n) \to f(x,y)$ so $f(x,y) = 0$.

6. Let (M,ρ) be $\{\frac{1}{n} : n \in \mathbb{N}\}$. Since $\frac{1}{n} \to 0$ (in $\mathbb{R}$) it is Cauchy – and that implies it is Cauchy in M, but if $\frac{1}{n} \to x \in M$, then $x = 0 \Rightarrow x \in M$ a contradiction.