Assignment 6

1. Theorem: Suppose that f is defined on $[a,b]$ and that $c \in (a,b)$ satisfies $f(c) = m$, $m = \inf\{f(x) : x \in [a,b]\}$. Then if f is differentiable at c, $f'(c) = 0$.

NOTE: This is more general than the book, and we don't have to use a proof by contradiction.

Pf: If $x > c$ then[+] $\dfrac{f(x)-f(c)}{x-c} \geq 0$, hence $\lim\limits_{\substack{x\to c\\ x>c}} \dfrac{f(x)-f(c)}{x-c} \geq 0$

If $x < c$ then[+] $\dfrac{f(x)-f(c)}{x-c} \leq 0$, hence $\lim\limits_{\substack{x\to c\\ x<c}} \dfrac{f(x)-f(c)}{x-c} \leq 0$.

Thus $f'(c) = \lim\limits_{\substack{x\to c\\ x\neq c}} \dfrac{f(x)-f(c)}{x-c}$. [+ because $f(x) \geq m$ for all x]

5. For any $a_1 \in (a,b]$ we have MVT $\Rightarrow$

$$\frac{f(a_1)-f(a)}{a_1 - a} = f'(c_1) \qquad c_1 \in (a, a_1)$$

hence $f'(c_1) = 0 \Rightarrow f(a_1) = f(a)$.

6. $$\frac{f(y)-f(x)}{y-x} = f'(z) \qquad z \in (x,y)$$

hence $f'(z) > 0 \Rightarrow f(y) - f(x) > 0 \Rightarrow f(y) > f(x)$

7. $\int_0^x f'(t)\,dt = f(x) - f(0) = f(x)$ + $\int_0^x g'(t)\,dt = g(x)$

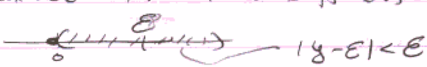
Thus $g(x) - f(x) = \int_0^x [g'(t) - f'(t)]\,dt \geq 0$

8. $|f(b) - f(a)| = \left|\int_a^b f'(x)\,dx\right| \leq \int_a^b |f'(x)|\,dx \leq \int_a^b (.3)\,dx$

hence $|f(b) - f(a)| \leq .3(b-a)$, i.e., $|f(b) - 2| \leq .3(b-a)$.

11(a) By assumption f'' is continuous. Let $\varepsilon = f''(c)$. Then we can choose $\delta > 0$ such that $|x-c| < \delta \Rightarrow |f''(x) - \varepsilon| < \varepsilon$

$\Rightarrow f''(x) > 0$.



b) f' is strictly increasing in $(c-\delta, c+\delta)$ [see 6 above] and $f'(c) = 0$.

c) f is decreasing in $(c-\delta, c)$ and increasing in $(c, c+\delta)$

12. Apply FTC to fg:

$$(fg)(b)-(fg)(a)=\int_a^b (fg)'(x)\,dx = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)$$

$$f(b)g(b) - f(a)g(a)$$

13. FTC(II): $\left[\int_0^x f(t)\,dt\right]' = f(x)$

Differentiate both sides of formula in 13:

$$f'(x) = 0 + 2f(x)$$

$$f'(x) - 2f(x) = 0 \qquad \frac{dy}{dx} - 2y = 0$$

$$\Rightarrow f(x) = Ce^{2x}$$

But from formula:

$$f(0) = 5 + 2\int_0^0 f(t)\,dt = 5$$

hence $Ce^{2\cdot 0} = 5 \Rightarrow C = 5 \Rightarrow f(x) = 5e^{2x}$.