Assignment 3

p.93:1 Let P: $a = x_0 < x_1 < \dots < x_n = b$

For each i, $[x_{i-1}, x_i] \cap \mathbb{Q} \neq \emptyset$

$[x_{i-1}, x_i] \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$ x_{i-1} x_i

(why? — you can prove this from decimal expansions)

Let $x_i' \in [x_{i-1}, x_i] \cap \mathbb{Q}$ $x_i'' \in [x_{i-1}, x_i]$

$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \geq f(x_i') = 1$

$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \leq f(x_i'') = 0$

Since $0 \leq f \leq 1$, $m_i = 0$, $M_i = 1$ and

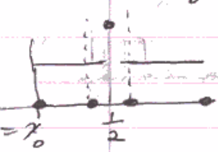
$$L_P(f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 0(x_i - x_{i-1}) = 0$$

$$U_P(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 1(x_i - x_{i-1}) = b - a.$$

$0 = \sup L_P(f) \neq \inf U_P(f) = 1 \Rightarrow f$ not R. integ.

p.94:2 Given $0 < \delta < \frac{1}{4}$, let

P_δ: $0 = x_0^\delta < x_1^\delta < x_2^\delta < x_3^\delta = 1$ $x_1 = \frac{1}{2} - \delta$, $x_2 = \frac{1}{2} + \delta$,



Then $m_1^\delta = m_2^\delta = m_3^\delta = 1$

$M_1^\delta = 1$, $M_2^\delta = 2$, $M_3^\delta = 1$

$$L_{P_\delta}(f) = 1\left(\left(\tfrac{1}{2} - \delta\right) - 0\right) + 1\left(\left(\tfrac{1}{2} + \delta\right) - \left(\tfrac{1}{2} - \delta\right)\right) + 1\left(1 - \left(\tfrac{1}{2} + \delta\right)\right) = 1$$

$$U_{P_\delta}(f) = 1\left(\left(\tfrac{1}{2} - \delta\right) - 0\right) + 2\left(\left(\tfrac{1}{2} + \delta\right) - \left(\tfrac{1}{2} - \delta\right)\right) + 1\left(1 - \left(\tfrac{1}{2} + \delta\right)\right) = 1 + 2$$

$|U_{P_\delta}(f) - L_{P_\delta}(f)| < 2\delta$ Let $\delta = \frac{\varepsilon}{2}$, we have

$$\int_0^1 f(x)\,dx = \lim L_{P_{\frac{1}{n}}}(f) = \lim (1) = 1.$$

3. a) $f(x) \geq 0 \Rightarrow m_i \geq 0 \Rightarrow L_P(f) = \sum m_i(x_i - x_{i-1}) \geq \sum 0(x_i - x_{i-1}) = 0$

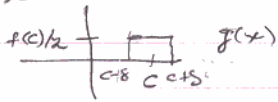
$$\Rightarrow \int_a^b f(x)\,dx \geq L_P(f) \geq 0.$$

b) Proof by contradiction. Say that $c \in [a,b]$, and $f(c) > 0$.

Choose $\delta > 0$ so that $|x-c| < \delta \Rightarrow |f(x) - f(c)| < f(c)/2$.

Then $f(c) - f(x) \le f(c)/2 \Rightarrow f(x) \ge f(c)/2$.

Let $g(x) = \begin{cases} f(c)/2 & |x-c| \le \delta \\ 0 & |x-c| > \delta \end{cases}$



Then $g(x)$ is R integrable and

$$\int_0^1 g(x)\,dx = \int_{x-\delta}^{x+\delta} f(c)/2\,dx = \frac{f(c)}{2}\cdot 2\delta = f(c)\delta$$

But $f(x) \ge g(x)$ for all $x \Rightarrow \int_0^1 f(x)\,dx \ge f(c)\delta > 0$.

c) Let $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$. We showed (in class) $\int_0^1 f(x)\,dx = 0$

even though $f(x)$ is not identically zero.

4 a) Let P_n: $x_0 = 0, \frac{1}{n}, \frac{2}{n}, \dots, x_i = \frac{i}{n}, \dots, x_n = 1$.

$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} = x_i$ $[f(x) = x]$

$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\} = x_{i-1}$

$$\Rightarrow U_{P_n}(f) - L_{P_n}(f) = \sum_{i=1}^{n} (x_i - x_{i-1})(x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} 1 = \frac{1}{n}$$

Choose n: $\frac{1}{n} < \varepsilon$.

b) $U_{P_n}(f) \longrightarrow \int_0^1 x\,dx$

$$U_{P_n} = \sum x_i(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i}{n}\left(\frac{i}{n} - \frac{(i-1)}{n}\right) = \sum \frac{i}{n}\cdot\frac{1}{n}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{1+\frac{1}{n}}{2}$$

$$\lim_{n\to\infty} U_{P_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2} \quad (= \lim L_{P_n}(f))$$

7. $\delta = .5$ $x_0 = 1, x_1 = 1.5, x_2 = 2$ because f is increasing

$f(x) = x^2$ $m_1 = \inf\{f(x) : 1 \le x \le 1.5\} = f(1) = 1$

$m_2 = \inf\{f(x) : 1.5 \le x \le 2\} = f(1.5) = 2.25$

$L_P(f) = 1(\frac{1}{2}) + (2.25)(\frac{1}{2}) = \dots$

$M_1 = f(1.5) = 2.25$ $U_P(f) = (2.25)(\frac{1}{2}) + 4(\frac{1}{2})$

$M_2 = f(2) = 4$