Assignment 4

p. 79: 3 Proof 1. We have

$$||f(x)|-|f(c)|| \le |f(x)-f(c)|$$

Choose $\delta>0$ so that $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$.

Then $|x-c|<\delta \Rightarrow ||f(x)|-|f(c)||<\varepsilon$.

Proof 2: $g(y)=|y|$ is cont because

$$|g(y)-g(c)| = ||y|-|c|| \le |y-c| \text{ etc.}$$

Thus the composition $|f(x)| = g(f(x))$ is cont.

5. Given $x\in\mathbb{R}$, let $q_n\in\mathbb{Q}$, $q_n\to x$. Then $0=f(q_n)\to f(x) \Rightarrow f(x)=0$

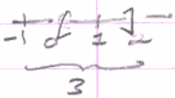
6. Say $x_n\in\mathbb{N}$ and $x_n\to c\in\mathbb{N}$. Choose N such that $n\ge N\Rightarrow |x_n-c|<1$. Then $x_n=c$ (why?). It follows that $n\ge N \Rightarrow f(x_n)=f(c)$ so $f(x_n)\to f(c)$

7. $$|f(x)-2| = |3x-1-2| = 3|x-1|$$

If you want $|f(x)-2|<\varepsilon$, you have to take $|x-1|<\frac{\varepsilon}{3}$.

8. $$|f(x)-1| = |x^2-1| = |x+1|\,|x-1|$$

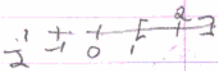
a) If $|x-1|\le 1$ then $|x+1| = |x-(-1)| \le 3$ (see picture) and



$$|f(x)-1|\le 3|x-1|.$$

Let $\delta=\min\{1,\varepsilon/3\}$. Then $|x-1|\le \delta \Rightarrow |f(x)-1|\le\varepsilon$

b) $$|f(x)-4| = |x^2-4| = |x+2|\,|x-2|$$



If $|x-2|\le 1$ then $|x-(-2)|\le 5 \Rightarrow |f(x)-4| \le 5|x-2|$.

Let $\delta=\min\{1,\varepsilon/5\}$.

c) We'll discuss this in class.

p. 86 1. $f(-3) = -27+12+2<0$, $f(0)=2$ $\Big\} \Rightarrow$ $f(c)=0$ for some $c\in[-3,0]$. Int. Value Thm

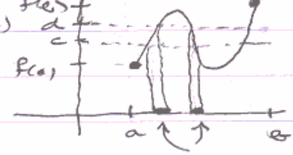
2 Let $f(x)=x^3+ax^2+bx+c$. Then

$$f(x) = x^3\left(1+a\frac{1}{x}+\frac{b}{x^2}+\frac{c}{x^3}\right)$$

$$\Rightarrow \lim_{x\to+\infty} f(x)=+\infty,\quad \lim_{x\to-\infty} f(x)=-\infty$$

$\Rightarrow \exists\ a<b$: $f(a)<0$, $f(b)>0$. Use Int. Value Thm.

3. We proved that f must assume its minimum value, i.e., $\exists$ $c\in[a,b]$ such that $f(c)\le f(x)$ for all $x\in[a,b]$. Let $\alpha=f(c)/2$.

6. a) 

$S=\{x: c<f(x)<d\}$

b) We have to show that if $x_1, x_2 \in S$ and $x_1<\bar{x}<x_2$ then $\bar{x}\in S$. We have $f(x_1)\le f(\bar{x})\le f(x_2)$ [f is increasing] hence $c\le f(x_1)\le f(x_2)<d$ implies that $c\le f(\bar{x})\le d$ so $\bar{x}\in S$.

→ see text!

7. By definition, there is an M such that for any $x, x'\in S$, $|f(x)-f(x')|\le M|x-x'|$. Given $\varepsilon>0$, let $\delta=\varepsilon/M$. Then $|x-x'|<\delta \Rightarrow |f(x)-f(x')|<\varepsilon$.

9. $f(x)=\frac{1}{x}$ is not unif cont because consider $x_n=\frac{1}{n}$ and $x_n'=\frac{1}{2n}$. We have $|x_n-x_n'|=\frac{1}{2n}\to 0$ but $|f(x_n)-f(x_n')|=|n-2n|=n\to\infty$. For any $x,x'\in[\mu,\infty)$

$$\left|\frac{1}{x}-\frac{1}{x'}\right| = \frac{|x-x'|}{|x||x'|}\le\frac{|x-x'|}{\mu^2}$$

hence given $\varepsilon>0$, let $\delta=\varepsilon\mu^2$. If $|x-x'|<\delta$, then $|f(x)-f(x')|<\varepsilon$

10. We did this in lecture! Recall: we showed f unif cont on $[1,\infty)$. Also f continuous on $[0,1]$ (the proof for $c\in(0,1]$ and for $c=0$ are different), hence f is uniformly cont on $[0,1]$.
f is unif cont on $[0,\infty)$: given $\varepsilon>0$ choose $\delta_1>0$, $\delta_2>0$ such that $|x-x'|<\delta_1 \Rightarrow |f(x)-f(x')|<\varepsilon/2$ for $x,x'\in[0,1]$
and $|x-x'|<\delta_2 \Rightarrow |f(x)-f(x')|<\varepsilon/2$ for $x,x'\in[1,\infty)$
Let $\delta=\min\{\delta_1,\delta_2\}$. If $x,x'\in[0,1]$,

$$|f(x)-f(x')|<\varepsilon/2<\varepsilon$$

If $x,x'\in[1,\infty)$

$$|f(x)-f(x')|<\varepsilon/2<\varepsilon$$

If $x<1<x'$, then $|x-x'|<\delta \Rightarrow$

$$|f(x)-f(x')|\le|f(x)-1|+|1-f(x')|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$