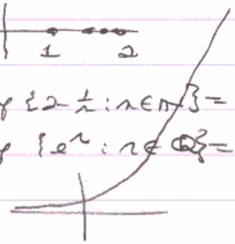
Assignment 3

1. p.54:1 $\sup\{\frac{1}{n}: n\in\mathbb{N}\} = 1$ $\inf\{\frac{1}{n}: n\in\mathbb{N}\} = 0$



$\sup\{2-\frac{1}{n}: n\in\mathbb{N}\} = 2$ $\inf\{2-\frac{1}{n}: n\in\mathbb{N}\} = 1$

$\sup\{e^r : r\in\mathbb{Q}\} = \infty$ $\inf\{e^r : r\in\mathbb{Q}\} = 0$

— look at graph of e^x

$\sup\{x^2 : 0\le x<2\} = 4$

$\inf\{x^2 : 0\le x<2\} = 0$

2. We are given $S\le a_n$ and $a_n\to a$. If $s\in S$ then $s\le a_n$ for all n implies that $s\le a$ [one of the elementary properties of limits]. Thus $S\le a$, and a is an upper bd.

2a) $\lim\limits_{n\to\infty}\frac{n}{n+1} = \lim\limits_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty}(1+\frac{1}{n})} = \frac{1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}\frac{1}{n}}$

$= \frac{1}{1+0} = 1$

b) $\lim\limits_{n\to\infty}\sqrt{n+1}-\sqrt{n} = \lim\limits_{n\to\infty}\frac{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}$

$= \lim\limits_{n\to\infty}\frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$

We have that $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} \overset{\circledast}{=} 0$ because given $\varepsilon>0$, choose N such

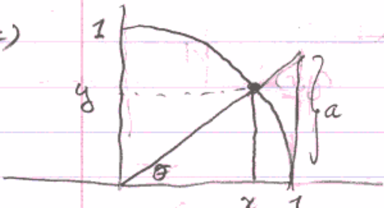
$\circledast$ uses Corollary 0.5

that $n\ge N \Rightarrow \frac{1}{n}<\varepsilon^2$. Then $\frac{1}{\sqrt{n}}<\varepsilon$. We have

$0 < \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} \to 0$

$\Rightarrow \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$

c)



$x=\cos\theta$ $y=\sin\theta$ $a=\tan\theta=\frac{\sin\theta}{\cos\theta}$

area small Δ < area $\triangle$ < area large Δ

$\frac{1}{2}\cos\theta\sin\theta < \frac{\theta}{2\pi}\cdot\pi < \frac{1}{2}\frac{\sin\theta}{\cos\theta}$

$$\Rightarrow \cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \Rightarrow \frac{1}{\cos\theta} > \frac{\sin\theta}{\theta} > \cos\theta$$

<u>CALCULUS</u>: $\lim_{\theta\to 0} \cos\theta = 1$ $\lim_{\theta\to 0} \frac{1}{\cos\theta} = 1 \Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

Let $\theta = \frac{1}{n}$: $\lim \frac{\cos\frac{1}{n}}{\frac{1}{n}} = 1,$

3. $a_n \in \mathbb{N}$ $a_n \to L \Rightarrow \exists N: n \geq N \Rightarrow L = a_N.$
Because choose N such that $n \geq N \Rightarrow |a_n - L| < \frac{1}{2}.$
Then $m, n \geq N \Rightarrow |a_n - a_m| < 1 \Rightarrow a_n = a_m$ (distinct integers have distance ≥ 1 from each other). Thus $a_n = a_N$ for all $n \geq N$, and since $|a_n - L| \to 0$, $L = a_N$.

4. This should have read "the sequence r_n given by $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}, \ldots$
If $x \in (0,1)$, let $x = .a_1 a_2 a_3 \ldots$ (decimal expansion)
Let $x_1 = .a_1$ $x_2 = .a_1 a_2$, $x_k = .a_1 \ldots a_k$.
We have $x_k \to x$. Any fraction r with $0 < r < 1$
occurs in the given sequence infinitely often (e.g., $\frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \ldots$
Let n_1 be the first integer with $x_1 = r_{n_1}$
Let n_2 " " first integer with $n_2 > n_1$ and $x_2 = r_{n_2}$
We have $r_{n_k} \to x$. If $x = 0$, then let
$n_1 < n_2 < \ldots$ be such that $r_{n_1} = 1, r_{n_2} = \frac{1}{2}, r_{n_3} = \frac{1}{3}, \ldots$
We have $r_{n_k} = \frac{1}{k} \to 0$.
Similarly if $r_{n_k} = \frac{k}{k+1}$, then $r_{n_k} \to 1$.
Since $0 \leq \{r_n\} \leq 1$, we see that the limits of subsequences is the set $[0,1]$.

5. a) $a = \sqrt{a^2} < \sqrt{2a}$, $\sqrt{2a} < \sqrt{2\cdot 2} = 2$
b) Let $a_1 = \sqrt{2}$, $a_2 = \sqrt{2a_1}$, $a_3 = \sqrt{2a_2}, \ldots$
Applying a) repeatedly (or using induction)
$\sqrt{2} \leq a_1 < a_2 < a_3 < \ldots < 2.$
Thus a_n converges to $b_0 = \sup \{a_n\}$.
c) We have that $a_n \to b_0$ and $a_{n+1} \to b_0$.
But $a_{n+1} = \sqrt{2a_n} \Rightarrow a_{n+1}^2 = 2a_n$
$a_{n+1}^2 \to b_0^2$, $2a_n \to 2b_0 \Rightarrow b_0^2 = 2b_0$ and $0 < b_0$
$(b_0 - 2)b_0 = 0 \Rightarrow b_0 = 2$

6.51.7 Given any m, n say $m < n$ we have

$$|a_n - a_m| = |(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{m+1} - a_m)|$$

$$\leq |a_n - a_{n-1}| + |a_{n-1} + a_{n-2}| + \cdots + |a_{m+1} - a_m|$$

$$\leq 2^{-(n-1)} + 2^{-(n-2)} + \cdots + 2^{-m}$$ (turn around

$$< 2^{-m} + 2^{-m-1} + 2^{-m-2} + \cdots$$

$$= \frac{2^{-m}}{1 - \frac{1}{2}}$$ (geometric series)

$$= 2^{-(m-1)}.$$

Given $\varepsilon > 0$ choose N such that $2^{-(N-1)} < \varepsilon$. Then

$m, n \geq N \Rightarrow$

$$|a_m - a_n| < 2^{-(m-1)} \leq 2^{-(N-1)} < \varepsilon.$$

1) b) $a_n^2 \leq \frac{n\,2}{n(n-1)} = \frac{2}{n-1}$

$\Rightarrow a_n \leq \frac{\sqrt{2}}{\sqrt{n-1}} \to 0$ (see 2 b) above).

7. §54: 5 We did this in lecture

6 We'll cover this in a future class.