Assignment 2

1 a) True

b) false (doesn't matter whether or not 1955 was bad year)

c) true

d) false

e) false

1.3.1. $f(n) = n/5$ defines 1-1 onto function

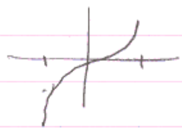
2. $f(n) = e^n$ defines 1-1 onto function $f: \mathbb{Z} \to S = \{e^n\}$.

We have proved $\mathbb{Z}$ is countable, so same is true for S.

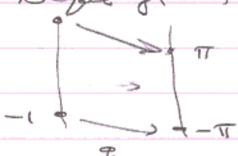
7. There are many solutions — here is one.

Define $f: (-\pi, \pi) \longrightarrow \mathbb{R}$ $f(x) = \tan^{-1}x$

f is 1-1 because $f'(x) = \frac{1}{1+x^2} > 0$. To see it is onto look at its graph.



Define $g: (-1, 1) \longrightarrow (-\pi, \pi)$ 1-1 onto 1



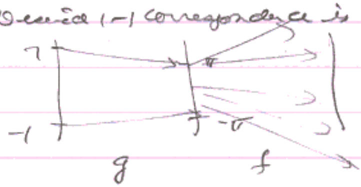
Find A and B:

$g(x) = Ax + B$

$g(-1) = -\pi$ $g(1) = \pi$

(you solve for A, B)

Desired 1-1 correspondence is $f \circ g$.



9a) Let $X = \{1, \dots, m\}$ $Y = \{1, \dots, n\}$

f	
1	n choices
2	n choices
3	⋮
⋮	
m	

n^m functions

b) $f: \{1, \dots, m\} \longrightarrow \mathbb{N}$ are in 1-1 correspondence with m-tuples $\{k_1, \dots, k_m\}$ $k_i \in \mathbb{N}$

$\mathcal{F}$ = set of fns $\mathcal{F} \approx \mathbb{N}^m$ which is countable.

c) $f: \mathbb{N} \longrightarrow \{1,\dots,n\}$ $\quad n>1$

are just the sequences $f(1), f(2), f(3), \dots$

where $f(k) \in \{1,\dots,n\}$. The diagonal argument shows uncountable — if there is a sequence $f_1, f_2, f_3, \dots$ we can list them:

$f_1(1), f_1(2), f_1(3), \dots$

$f_2(1), f_2(2), f_2(3), \dots$

$f_3(1)\ f_3(2)\ f_3(3), \dots$

Let $': \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ be 1-1 with $n' \neq n$. Let g be given by $g(k) = f_k(k)'$. g isn't in the list

d) Let S be all finite subsets $\{n_1, n_2, \dots, n_r\}$ of $\mathbb{N}$

Let P be all finite n-tuples $(n_1, \dots, n_r)$ (r arbitrary)

Define $\theta: P \longrightarrow \mathbb{N} : (n_1, \dots, n_r) \mapsto 2^{n_1} \cdots p_r^{n_r}$

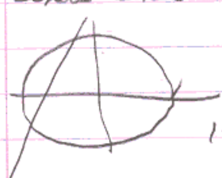
This is an injection (1-1) so P is countable.

Define $g: P \longrightarrow S$ by $g((n_1, \dots, n_r)) = \{n_1, \dots, n_r\}$.

Then g is onto, so S is countable.

1.4.5 Say that $x^2 + y^2 \leq 1$. If in addition $\frac{x}{2} + \frac{y}{3} = 1$

then that line intersects the disk $\Rightarrow$ it intersects the circle, i.e., $x^2 + y^2 = 1$. Find intersection:



$$y = 3\left(1 - \frac{x}{2}\right)$$

$$1 = x^2 + 9\left(1 - \frac{x}{2}\right)^2 = x^2 + \left(\frac{x^2}{4} - x + 1\right)9$$

$$\frac{9x^2}{4} + x^2 - 9x + 9 = 1$$

$$\frac{13x^2}{4} - 9x + 8 = 0$$

imaginary

$$x = +\frac{9 \pm \sqrt{81 - 13 \cdot 8}}{2 \cdot \frac{13}{4}}$$ contradiction

6. $n=1$ $\quad 1^3+5\cdot 1 = 6$ is divisible by 6

Suppose $P(n)$ is true i.e. n^3+5n+1 is divisible by 6.

Then

$$(n+1)^3+5(n+1)+1$$
$$= n^3+3n^2+3n+1+5n+5+1$$
$$= (n^3+5n+1)+(3n^2+3n+6)$$
$$= (n^3+5n+1)+3(n+1)n+6$$

The first and last terms are divisible by 6.

But $n(n+1)$ is always even (either n or $n+1$ is even)

hence $3(n+1)n$ is also divisible by 6.

8. $1^3 = 1^2$

Say formula true for n. Then

$$((1+2+\dots+n)+(n+1))^2 = (1+\dots+n)^2+2(1+\dots+n)(n+1)$$
$$+(n+1)^2$$

But we have $1+\dots+n = \dfrac{n(n+1)}{2}$ (arithmetic series).

$$= 1^3+\dots+n^3+ \quad n(n+1)(n+1)+(n+1)^2$$
$$= 1^3+\dots+n^3+(n+1)^3$$

9. $1 \geq 1$

Say inequality true for n.

$$(1+x)^n \geq 1+nx$$

$$\Rightarrow (1+x)(1+x)^n \geq (1+x)(1+nx) \qquad (\text{since } \underline{1+x\geq 0})$$
$$= 1+x+nx+nx^2$$
$$\geq 1+(n+1)x \qquad \text{since } \underline{nx^2\geq 0}.$$