HOPF ALGEBRAS, INTERVAL PARTITIONS, AND NON-COMMUTATIVE LAGRANGE INVERSION

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ABSTRACT. The Hopf algebra introduced by Brouder, Fabretti, and Krattenaler in the context of non-commutative Lagrange inversion can be identified with the reverse of the incidence algebra of interval partitions. The reverse antipode determines the (generally distinct) left and right inverses of power series with non-commuting coefficients and non-commuting variables. The summands of the reversed antipode are indexed by reduced planar trees. Replacing depth first ordering with breadth first ordering, the summands of the antipode are indexed with irreducible (non-order contractible) trees, in which precisely one multiple parent vertex occurs at each level.

1. INTRODUCTION

Non-commutative power series play an important role in a number of areas, including combinatorics, free probability, and quantum field theory. A striking aspect of this work is that one can effectively manipulate series in which neither the coefficients nor the variables commute. Such calculations can often be simplified through the use of combinatorial indices such as trees and graphs. In turn, these somewhat *ad hoc* techniques can frequently be systematized by using Hopf algebras. This approach was pioneered by Rota and his colleagues in their studies of combinatorics [6]. More recently Kreimer [8], and Kreimer and Connes [2] have used Hopf-theoretic methods to rationalize various Feynman diagram methods used in perturbative quantum field theory.

An important example of these technques was described by Haiman and Schmitt [5], who showed that calculating the antipode for the reduced Faà di Bruno Hopf algebra is equivalent to finding an explicit Lagrange inversion formula for factorial power series with commuting coefficients. For this purpose they realized the Hopf algebra as the incidence algebra of colored partitions of finite colored sets. In this context they used colored trees to index the terms in formulae for the antipode.

In a recent paper, Brouder, Fabretti, and Krattenaler [1] described a Hopf algebra, called the BFK algebra below, which is related to Lagrange inversion for power series with non-commuting coefficients. As they pointed out, the situation is more delicate, since the "composition" of such polynomials is not associative.

We begin by showing that the multivariable BFK algebra is essentialy dual to the incidence Hopf algebra determined by colored *ordered* (i.e., *interval*) partitions. An important distinction between this theory and that of Haimann and Schmitt on the Faa di Bruno algebra is that the antipode S does not satisfy $S^2 = I$ (This was pointed out in [1]). In fact S^{-1} provides the antipode for the *reversed* Hopf algebra, which coincides with the BFK Hopf algebra (see §6). An advantage of this abstract approach is that the associativity of the product is trivial, and the

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calculation of the antipode follows from an elementary formula for antipodes in arbitrary incidence algebras [10].

Our main purpose is to show that one can use *reduced planar colored trees* and their *depth first ordering* to find an efficient expression for the reversed antipode S^{-1}. This provides a surprising analogue of the Haiman-Schmitt expression for general partitions and reduced colored trees, but it is proved in a completely different manner. We show that dually one can use *ordered irreducible simple trees* and their *breadth first ordering* for indexing the summands of the antipode S.

In §10 we show that show that despite the fact that the substitution operation for non-commutative power series is not associative, one can still use the Hopf algebra (which is associative) to find the left and right substitutional inverses of power series in which neither the variables nor the constants commute. As we show, these inverses are generally distinct.

Finally we remark that Rota's Hopf incidence algebras can be regarded as "K_0 quantum groups" of suitable families of partially ordered sets. As in the K-theory of rings (see, e.g., [3] for a particularly simple example), one is interested in defining an algebraic object, such as a group or in this context "quantum group" (i.e., Hopf algebra), which is generated by "dimension" invariants. In both contexts one first assumes that the family is closed under something like Cartesian products to obtain large dimensions, and one deals with pairs or segments. To obtain the desired algebraic object, one divides out by "degenerate" elements. It is tempting to conjecture that there is also a corresponding "K_1 quantum group" for combinatorial lattices. This will be considered elsehwere. In any event, owing to this more general perspective, we expect that Hopf incidence algebras will arise in other areas of modern mathematics.

In order to make the material more accessible to functional analysts and mathematical physicists, we have included a careful exposition of the relevant constructions from algebraic combinatorics.

2. Ordered sets and their colorings

A *partially ordered set* $(P, \leq)$ is a set P together with a relation $\leq$ such that $x \leq y$ and $y \leq x$ if and only if $x = y$, and $x \leq y \leq z$ implies $x \leq z$. We say P is a *linearly ordered set* or simply an *ordered set* if $x \leq y$ or $y \leq x$ for all $x, y \in X$. Given $x, y \in P$ with $x \leq y$, we let $[x, y]$ denote the *segment* $\{z \in P : x \leq y \leq z\}$. We denote a finite ordered set S by $(x_1, \dots, x_p)$ where $x_1 < \cdots < x_p$. In particular if $p \in \mathbb{N}$, we let $[p] = (1, \dots, p)$. Given partially ordered sets P and Q, we let $P \times Q$ have the *product* partial ordering $(x_1, x_2) \leq (y_1, y_2)$ if $x_1 \leq y_1$ and $x_2 \leq y_2$.

We fix a set of "colors" $\Gamma = \{1, 2, 3, \dots, N\}$. A *colored ordered set* (S, γ) is an ordered set S together with a function $\gamma = \gamma_S : S \to \Gamma$ (we place no restrictions on c). We refer to $\gamma = \gamma(x)$ as the *color* of a point $x \in S$, and we say that γ is a *coloring* of S. If $S = (x_1, \dots, x_s)$ and $\gamma_i = c(\gamma_i)$, γ is determined by the word $v = \gamma_1 \cdots \gamma_p$ in the free monoid Γ^* generated by Γ. The identity e of this monoid is the empty word $\emptyset$. We write $|v|$ for the length of v. Two colored ordered sets (S, v) and (T, w) are *isomorphic* or *have the same coloring* if there is an order isomorphism of $\theta : S \to T$ that preserves the coloring, i.e., $\gamma_T(\theta(s)) = \gamma_S(s)$. Since these are linearly ordered sets, the mapping θ will necessarily be unique.

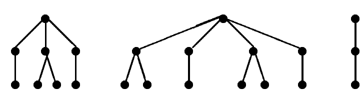
Given a finite colored ordered set S, any subset $R \subseteq S$ is itself totally ordered in the relative order, and we let R have the restricted coloring $\gamma|R$. Given disjoint colored sets S and T, with colorings v and w, we let $S \sqcup T$ denote the union with the left to right ordering, and the coloring $\gamma_{S \sqcup T} = vw$.

3. Planar trees and their colorings.

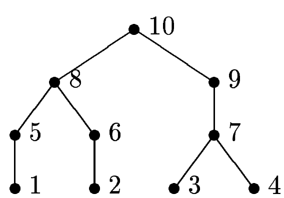
The *planar forests* are defined recursively. For transparency we use terms associated with botanical and genealogical trees. To construct a plane forest F we first chooses an ordered set F_1 of *vertices* $(x_1, \ldots, x_r)$ called the *roots* or the first *level* of F. For each root x_i in F we then choose a possibly empty ordered set of vertices $(x_{i1}, \ldots, x_{ir_i})$ called the *children* of x_i. The entire collection F_2 of these children is called the second level, which we totally order first by their parent and then among siblings by the given order. Having chosen the n-th level, we choose an ordered set of vertices for each vertex in that co-set. These new vertices consititute the $(n+1)$-st level, and we order them in the same manner. We only consider finite forests.

We define the n-th *layer* of F to be the forest obtained by considering only the vertices in the n-th and $(n+1)$-st levels together with the edges joining them in F. A *tree* is a forest with only one root. We call the vertices of a forest F without children the *leaves* of F. We say that a vertex is *degenerate* if it has precisely one child. Finally a non-degenerate vertex is *simple* if it is the only non-degenerate vertex on that level. We say that the corresponding level is simple, and that a tree is simple if all of its vertices are simple.

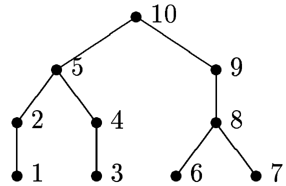
We may identify a forest with a graph in the plane in the usual manner. The levels are placed in horizontal rows, parents are joined by edges to their children, and the left to right order reflects the recursively defined order on the parents, and the given order on the children in each family. A typical planar forest is illustrated below:



We have a total *breadth first* ordering $\ll$ on the vertices of a planar tree T. Counting downwards (from the total number of vertices), we begin with the root of the tree and we then choose the rightmost vertex in the second level. We number the elements of that level from right to left in decreasing order, and then go to the rightmost vertex of the next level, and continue in this fashion. When all the vertices of the tree are numbered we move to the next tree on the level. In a given tree we write $x \ll y$ if $x \leq y$, or x is in the same level as x and lies to the left of y as illustrated in the following figure.



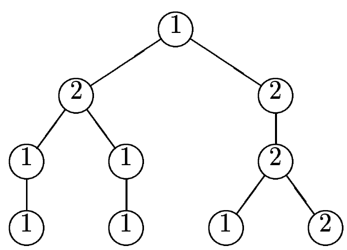
We will also use the *depth first* ordering of a tree. Again *counting downwards*, one first one starts at the rightmost root and then successively chooses right branches going to successive generations, and backtracking to the next right-most uncounted vertex. The resulting order $\lll$ has as its greatest vertex the root of the rightmost tree, as illustrated in the diagram below.



The depth first relation $x \lll y$ may be described in the following manner. For any two vertices x, y either there is a unique vertex with branches leading to both x and y.or one is a descendant of the other. We have $x \lll y$ if and only if the branch leading to x is to the left of that leading to y, or x is a descendant of y. We note that on each row the two total orderings coincide, i.e., $x \ll y$ if and only if $x \lll y$.

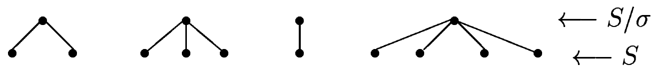
A forest F is said to be *reduced* if it has no degenerate vertices. On the other hand it is *layered* if at each level other than the last, each vertex has a child, and each non-leaf level has at least one non-degenerate vertex (hence the $(n+1)$-st level is larger than the n-th level).

A *colored forest* (F, γ) consists of a forest F together with a coloring of the vertices $\gamma : F \to \Gamma$ such that if x is a degenerate vertex with child y, then $\gamma(y) = \gamma(x)$. The following is a 2-colored layered tree.



4. Colored ordered partitions and their segments

An *ordered* (or *interval*) *partition* $\sigma = (B_1, \ldots, B_q)$ of S is a collection of subsets for which $\bigcup B_k = S$ and $B_1 < \ldots < B_q$ in the given "left to right" ordering. We may use the planar forest

(4.1) 

or the parenthetical expression

$$(12)(345)(6)(789\overline{10})$$

to denote the partition $\sigma = (B_1, \ldots, B_4)$ of $[10]$ with

$$B_1 = \{1, 2\}, B_2 = \{3, 4, 5\}, B_3 = \{6\}, B_4 = \{7, 8, 9, 10\}.$$

We regard a partition as an equivalence relation on S and identify the quotient set S/σ with $(B_1, \ldots, B_q)$. We say that a block is a *singleton* if it has only one element, and that its element is *degenerate*.

There is an alternative approach to partitions that is useful. We define an (abstract) partition σ of an ordered set S to be an increasing map $f_\sigma : S \to S_\sigma$ of

S onto an ordered set $T = S_\sigma$. We then have the ordered partition $(B_t)_{t \in T}$, where $B_t = f_\sigma^{-1}(t)$. Conversely given a partition $\sigma = (B_1, \dots, B_q)$ in our initial sense, we have a corresponding increasing surjection $f_\sigma : S \to S/\sigma = (B_1, \dots, B_q)$, where $f(x) = B_j$ if $x \in B_j$. (4.1) may be regarded as the mapping diagram of f_σ in that example.

Given a second ordered partition $\pi = (C_1, \dots, C_r)$ of S, we write $\sigma \preceq \pi$ if every set B_j is contained in some set C_k, i.e., σ is a *refinement* of π. Equivalently, $f_\pi = g \circ f_\sigma$ for some increasing function $g : S_\sigma \to S_\pi$. We write $\sigma \prec \pi$ if $\sigma \preceq \pi$ and $\sigma \neq \pi$. Given partitions σ of S and π of T, we have that

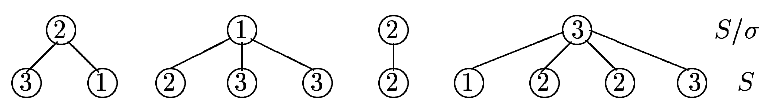
$$\sigma \sqcup \pi = \{B \sqcup C : B \in \sigma, C \in \pi\}$$

is an ordered partition of $S \sqcup T$ (we use the left to right ordering).

A *colored partition* (σ, γ_σ) of a colored ordered set (S, γ_S) is an ordered partition $\sigma = (B_1, \dots, B_q)$ of S together with a coloring c_σ of $(B_1, \dots, B_q)$, such that if $B_j = \{x\}$, then $c_\sigma(B_j) = c_S(x)$, i.e., singletons have the same color as their unique element. Equivalently, we have a colored ordered set S_σ and an order preserving surjection $f = f_\sigma : S \to S_\sigma$ with the property that if x is degenerate, then $c(f(x)) = c(x)$. Given colored ordered sets S and T, we say that a mapping $f : S \to T$ is *proper*, and write $f : S \twoheadrightarrow T$ if it has the latter properties, i.e., it is a proper surjection satisfying the singleton condition. We say that σ is a *partition of S with the coloring $w = c_\sigma$* and that $(S/\sigma, w)$ is a colored ordered set. We have that $(\sigma \sqcup \tau, vw)$ is a colored ordered partition of the colored ordered set $S \sqcup T$.

We use the colored planar forest (or layer, see above)

(4.2)



or the parenthetical expression

$$(31)_2(233)_1(2)_2(1223)_3$$

to indicate the colored ordered partition $\sigma = ((12), (345), (6), (789\overline{10}))$, $w = \gamma_\sigma = 2123$ of the colored set

$$([10], 3123321223)$$

In this example we have the colored order isomorphism

$$([10]/\sigma, w) \cong ([4], 2123).$$

If S is a colored ordered partition, we define $\mathcal{Y}(S)$ to be the collection of all colored ordered partitions σ of the colored set S. We partially order $\mathcal{Y}(S)$ by $(\sigma, v) \preceq (\pi, w)$ if (1) $\sigma \preceq \pi$, and (2) for any $B \in \sigma \cap \pi$, $\gamma_\sigma(B) = \gamma_\pi(B)$ (i.e., B has the same color in either partition). For simplicity we simply write $\sigma \preceq \pi$. Owing to the second condition, if $\sigma \preceq \pi$ and $\pi \preceq \sigma$, then $\sigma = \pi$ as colored sets. It is evident that $\sigma \preceq \pi$ if and only if $f_\pi = g \circ f_\sigma$, for a (necessarily unique) proper function $g : S_\sigma \twoheadrightarrow S_\pi$. If $S = (x_1, \dots, x_p)$, has the coloring $v = v(1) \cdots v(p)$, then $\mathcal{Y}(S)$ has the minimum element $\mathbf{0}_v = ((x_1) \dots (x_p), v)$ and the maximal elements $\mathbf{1}_j = ((x_1 \dots x_p), j)$ where $j \in \Gamma$.

We turn next to segments of colored partitions $P = [\sigma, \tau]$, where $\sigma \preceq \tau \in \mathcal{Y}(S)$. P has the relative partial ordering $\preceq$, and an element $\lambda \in P$ may be regarded as a colored ordered partition of the elements in S/σ . Given segments $P \subseteq \mathcal{Y}(S)$

and $Q \subseteq \mathcal{Y}(T)$, we say that P is *isomorphic* to Q, and write $P \simeq Q$, if there exists an order isomorphism $\theta : P \to Q$ such that for each $\lambda \in P$, S/λ and $T/\theta(\lambda)$ are isomorphic colored ordered sets. In particular, for any partitions σ and τ, the segments $P = [\sigma, \sigma]$ and $Q = [\tau, \tau]$ are equivalent if and only if S/σ and T/τ are isomorphic colored ordered sets.

Given a coloring v of $[p]$ and $j \in [p]$, we let $Y_v^j = [\mathbf{0}_v, \mathbf{1}_j]$. Given $\sigma \prec \pi$ and $\sigma' \prec \pi'$, we have an order isomorphism

$$\theta : [\sigma, \pi] \times [\sigma', \pi'] \simeq [\sigma \sqcup \sigma', \pi \sqcup \pi'],$$

where for each λ, $\theta((\lambda, \lambda'))$ and (λ, λ') are isomorphic colored ordered sets.

Lemma 1. *Let us suppose that (S, s) is a colored ordered set and that $(\sigma, v) \preceq (\pi, w)$ in $\mathcal{Y}(S)$. Then letting $v_k = v|C_k$ we have a natural equivalence*

$$\theta : [\sigma, \pi] \cong Y_{v_1}^{w(1)} \times \ldots \times Y_{v_q}^{w(q)}$$

where for each $\lambda \in [\sigma, \pi]$, λ and $\theta(\lambda) = (\lambda_1, \ldots, \lambda_k)$ are isomorphic colored sets.

Proof. Consider the mapping $g : S_\sigma \twoheadrightarrow S_\pi$ described above. Let us identify S_π with $[q]$. The intermediate colored partitions correspond to factorizations $S_\sigma \twoheadrightarrow T \twoheadrightarrow S_\pi$. To construct such a diagram it suffices to choose for each $j \in S_\pi$ a factorization $g^{-1}(j) \twoheadrightarrow T_j \twoheadrightarrow \{j\}$, i.e., an element λ_j of $Y_{v_j}^j$, where v_j is the coloring of the interval $g^{-1}(j)$. It is evident that we have a one-to-one order preserving correspondence $\lambda \leftrightarrow (\lambda_1, \ldots, \lambda_q)$ with the desired coloring property. ∎

In order to obtain the incidence Hopf algebras, it is necessary to impose a more inclusive equivalence relation which identifies all of the one block partitions with a multiplicative identity. Given a colored partition (σ, v) of S, we let S_{ns} be the union of the non-singleton blocks in σ, σ_{ns} be the collection of non-singleton blocks, and v_{ns} be the restriction of v to σ_{ns}. We say that order intervals $P = [\sigma, \tau] \subseteq \mathcal{Y}(S)$ and $Q = [\sigma', \tau'] \subseteq \mathcal{Y}(T)$ are *similar,* and write $P \sim Q$, if there is an order-preserving bijection $\theta : P \to Q$ such that for each $\lambda \in P$, λ_{ns} and $\theta(\lambda)_{ns}$ have the same coloring, i.e., S_{ns}/λ_{ns}. If the non-singleton sets S_{ns} and T_{ns} are empty, this is a vacuous restriction. For any colored partition σ and segment P,

$$[\sigma, \sigma] \times P \sim P.$$

To prove this we consider the mapping

$$\theta : [\sigma, \sigma] \times P \sim P : \sigma \sqcup \lambda \mapsto \lambda$$

This is clearly a bijection and order preserving. Since $(\sigma \sqcup \lambda)_{ns} = (\lambda)_{ns}$, θ satisfies the coloring condition. Similarly we have that $P \times [\sigma, \sigma] \sim P$ for any σ and P. Finally it is easy to verify that if $S \sim S'$ and $T \sim T'$, then $S \times T \sim S' \times T'$.

We let $\mathcal{P}_0$ denote the class of all segments $P = [\sigma, \pi] \subseteq \mathcal{Y}(S)$ for arbitrary finite colored sets S. It is evident that $\mathcal{P}_0$ is closed under Cartesian products and the formation of subintervals. We let $\mathcal{P} = \mathcal{P}_0/\sim$ be the set of similarity classes $[P]_\sim$ and we define a monoid operation on $\mathcal{S}$ by

$$[[\sigma, \tau]]_\sim [[\sigma', \tau']]_\sim = [[\sigma \times \sigma', \tau \times \tau']]_\sim.$$

The corresponding multiplicative identity is given by $1 = [Y_j^j]$, where j is arbitrary.

On the other hand the intervals Y_v^j with $|v| > 1$ are all non-similar, and we will simply write Y_v^j for their similarity classes $[Y_v^j]_\sim$ as well. When confusion is unlikely, we will dispense with the similarity class notation $[\ ]_\sim$ altogether. It is evident that $\mathcal{P}$ is just the free monoid on the symbols Y_v^j with $|v| > 1$.

5. Hopf algebras

We briefly recall some elementary notions from the theory of Hopf algebras. More complete discussions can be found in [7], [11], [4], and [9].

Given a vector space V, we let $I : V \to V$ denote the identity mapping, and $L(V)$ the algebra of all linear mappings $T : V \to V$. Given a unital algebra $(A, 1)$ the *tensor product algebra* $A \otimes A$ is given the associative multiplication

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2$$

and the multiplicative unit $1 \otimes 1$. A *bialgebra* $(H, m, \eta, \Delta, \varepsilon)$ consists of a vector space A with an associative product $m : H \otimes H \to H$, a homomorphism $\eta : \mathbb{C} \to H : \alpha \to \alpha 1$, where 1 is a multiplicative unit for H, a coassociative coproduct $\Delta : H \to H \otimes H,$: and a counit $\varepsilon : H \to \mathbb{C}$ with the linking property that $\Delta : H \to H \otimes H$ is a unital homomorphism. We employ Sweedler's notation

$$\Delta a = \sum a_{(1)} \otimes a_{(2)}.$$

An *antipode* for a bialgebra H is a mapping $S : H \to H$ such that for any $a \in H$

$$\sum S(a_{(1)})a_{(2)} = \sum a_{(1)} S(a_{(2)}) = \varepsilon(a)1.$$

or equivalently, $m(S \otimes I)\Delta = m(I \otimes S)\Delta = \eta \circ \varepsilon$. We say that S is a *left antipode* if one just has the first an third terms are equal, and a *right antipode*, if one has the second equality. If H has an antipode, then any left (respectively right) antipode automatically coincides with S, and in particular, S is unique. An antipode S is automatically a unital antihomorphism, i.e., we have

$$\begin{aligned} S(gh) &= S(h)S(g) \\ S(1) &= 1 \end{aligned}$$

(see [11] Prop. 4.0.1). A Hopf algebra $(H, m, \eta, \Delta, \varepsilon, S)$ is a bialgebra $(H, m, \eta, \Delta, \varepsilon)$ together with an antipode S. Given a Hopf algebra H with antipode S, we have that $S^2 = I$ if and only if

$$\sum S(a_{(2)})a_{(1)} = \sum a_{(2)} S(a_{(1)}) = \varepsilon(a)1 \tag{5.1}$$

(see [7], Th. III.3.4).

Given $\varphi, \psi \in L(H)$, we define the *convolution* $\varphi * \psi \in L(H)$ by

$$\varphi * \psi(x) = \sum \varphi(x_{(1)})\psi(x_{(2)})$$

or equivalently, $\varphi * \psi = m \circ (\varphi \otimes \psi) \circ \Delta$. This determines an associative product on $L(H)$ with the multiplicative unit $u = \eta \circ \varepsilon$. It is evident form the definition that S is an antipode if and only if $S * I = I * S = u$, i.e., it is a convolution inverse of I.

A bialgebra $(H, m, \eta, \Delta, \varepsilon, S)$ is said to be *filtered* if one has an increasing sequence of subspaces H_n $(n \geq 0)$ with $\bigcup H_n = H$, for which

$$\begin{aligned} H^m H^n &\subseteq H^{m+n} \\ \Delta H^n &\subseteq \sum_{p+q=n} H^p \otimes H^w. \end{aligned}$$

and it is said to be *connected* if in addition $H^0 = \mathbb{C}1$. We say that H is *graded* if there are subspaces H_n of H with $H_n \cap H_m = \{0\}$ and $\sum H_n = H$ such that

$$\begin{aligned} H_m H_n &\subseteq H_{m+n} \\ \Delta H_n &\subseteq \sum_{p+q=n} H_p \otimes H_q. \end{aligned}$$

It is easy to see that such a system determines a filtration if we let $H^n = \sum_{i=0}^n H_i$.

If H is a connected filtered Hopf algebra, then for any $a \in H^n$,

$$\Delta a = a \otimes 1 + 1 \otimes a + \sum \tilde{a}_{(1)} \otimes \tilde{a}_{(2)}$$

where $\rho(\tilde{a}_{(1)}) + \rho(\tilde{a}_{(2)}) = n$ and $\rho(\tilde{a}_{(1)}), \rho\tilde{a}(_{(2)}) > 0$. In particular if $x \in H_1$, it follows that

$$\Delta x = x \otimes 1 + 1 \otimes x.$$

This is used to prove the following result:

Theorem 2. *If H is a connected filtered bialgebra, then it has an antipode given by the "geometric series"*

$$Sa = (u - (u - I))^{-1}(a) = \sum_{k=0}^{\infty} (u - I)^{*k} a. \tag{5.2}$$

The sum is finite for each a since if $a \in H^n$, then $(u - I)^{*(n+1)}(a) = 0$.

Assuming that H is a connected graded Hopf algebra let us suppose that $a \in H_n$, $n > 0$ Then from [4] or [9]

$$\Delta a = a \otimes 1 + 1 \otimes a + \sum_{1}^{n-1} a_k \otimes b_{n-k}$$

where $a_k \in H_k$, $b_{n-k} \in H_{n-k}$, and thus

$$0 = \varepsilon(a)1 = S(a) + a + \sum S(a_k) b_{n-k}$$

S is thus recursively determined by $S(1) = 1$ and if $a \in H_n$ with $\varepsilon(a) = 0$, then

$$S(a) = -a - \sum_{k=1}^{n-1} S(a_k) b_{n-k}.$$

It immediately follows from this relation that $S(H_n) \subseteq H_n$.

Proposition 3. *If H is a connected graded Hopf algebra and each subspace H_n is finite dimensional, then $S : H \to H$ is a bijection.*

Proof. This follows from (5) by induction on the grading. Since $S(1) = 1$ and $H_0 = \mathbb{C}1$, $S(H_0) = H_0$. Let us suppose that $S(H_k) = H_k$ for each $k \leq n - 1$. Given $a \in H_n$, $n > 0$, we have that

$$a = -S(a) - \sum_{k=1}^{n-1} S(a_k) b_{n-k}.$$

By induction we may assume that $b_{n-k} = S(c_{n-k})$ for some $c_{n-k} \in H_{n-k}$. It follows that

$$a = -S(a) - \sum_{k=1}^{n-1} S(a_k) S(c_{n-k}) = S(-a - \sum c_{n-k} a_k)$$

where $-a-\sum c_{n-k}a_k \in H_n$. ∎

Given any vector space V, we define the *flip* $\tau : V\otimes V \to V\otimes V$ by $\tau(v\otimes w) = w\otimes v$. Given a bialgebra $H=(H,m,1,\Delta,\varepsilon)$ we define the *reversed* bialgebra to be $H^\tau=(H,m,1,\Delta^\tau,\varepsilon)$ (this is indeed another bialgebra: see [7]). It is shown in [7], Cor. III.3.5 that if the antipode S of a Hopf algebra H is invertible, then S^{-1} is an antipode for the reversed algebra H^τ. In particular since

$$\Delta^\tau a = \sum a_{(2)}\otimes a_{(1)}.$$

it follows that S^{-1} is characterized by the relation

$$\sum S^{-1}(a_{(2)})a_{(1)} = \sum a_{(2)}S^{-1}(a_{(1)}) = \varepsilon(a)1.$$

It is recursively determined by $S^{-1}(1)=1$ and if $a\in H_n$ (hence $\varepsilon(a)=0$,) then

$$S^{-1}(a) = -a - \sum_{k=1}^{n-1} b_k S^{-1}(a_{n-k}). \tag{5.3}$$

6. The Lagrange Hopf incidence algebra

We define the *(colored) Lagrange incidence Hopf algebra* $\mathcal{L}$ as follows. As an algebra, $\mathcal{L}$ is the free associative unital algebra generated by the segments Y^i_u $(1\le i\le N, |u|\ge 2)$ with unit 1. We may regard the elements of $\mathcal{L}$ as linear combinations of elements of the free monoid $\mathcal{P}$ of intervals $P=Y^{w(1)}_{u_1}\times\ldots\times Y^{w(q)}_{u_q}$, where $|u_k|\ge 2$. Following Joni and Rota [6], p. 98, the coproduct

$$\Delta : \mathcal{L}\to\mathcal{L}\otimes\mathcal{L}$$

is defined by

$$\Delta(P) = \sum_{\pi\in Y^i_v} [0_P,\pi]\otimes[\pi,1_P],$$

and the coidentity homomorphism $\varepsilon:\mathcal{L}\to\mathbb{C}$ is determined by $\varepsilon(e)=1$ and $\varepsilon(Y^i_v)=0$ for v with $|u|\ge 2$.

Given $(\pi,w)\in Y^i_u$ with $w\ne 0_u, 1_i$we have that $\pi=(C_1,\ldots,C_q)$ is an ordered partition of the colored set $([p],u)$ where $p=|u|$, and $w=w(1)\cdots w(q)$ is a coloring of π. We let $u_i=u|C_i$. From above we have that $[0_u,\pi]\sim Y^{w(1)}_{u_1}\ldots Y^{w(s)}_{u_s}$ and $[\pi,1_i]$ $=[0_w,1^i]=Y^i_w$. On the other hand if $\pi=0_P$, $[0_P,\pi]=1$, whereas if $\pi=1_P$, then $[\pi,1_P]=1$, from which we conclude that

$$\Delta(Y^i_u) = Y^i_u\otimes 1 + 1\otimes Y^i_u + \sum_{q=1}^{p}\sum_{\pi\in\mathcal{I}_q(p)}\sum_{w\in[N]^q} Y^{w(1)}_{u_1}\ldots Y^{w(q)}_{u_q}\otimes Y^i_w. \tag{6.1}$$

Given a segment $P=[\sigma,\tau]$ in Y^i_v we define a *chain* γ in P to be a sequence $\sigma=\sigma_0\prec\sigma_1\prec\ldots\prec\sigma_r=\tau$. Given such a chain, each interval $[\sigma_{k-1},\sigma_k]$ determines the k-th layer of a layered forest $F(\gamma)$, with r layers (and thus $r+1$ levels) and conversely a layered tree with r layers wilI determine a chain in P. If $P=Y^i_u$, any chain γ can be extended to a chain γ' such that for each k,σ'_{k-1} is obtained from σ'_k by splitting one of the non-singleton blocks of σ_k into a singleton and another block. The corresponding trees are simple and each non-degenerate vertex has two children. Examining the tree of such a chain it follows that the maximal chains in

Y_u^i all have $|u|-1$ elements, and thus $\rho(Y_u^i)=|u|-1$. Given an arbitrary segment $P=[\sigma,\tau]\cong Y_{v_1}^{w(1)}\times\ldots\times Y_{v_q}^{w(q)}$, it is evident that

$$\rho(P)=\rho(Y_{v_1}^{w(1)})+\ldots+\rho(Y_{v_q}^{w(q)}).$$

We define $\mathcal{L}_n$ to be the linear subspace spanned by the intervals P with $\rho(P)=n$.

We note that given a generator Y_u^j, a partition $\pi=(C_1,\ldots,C_q)$ of $p=|u|$, and $w\in[N]^q$, s

$$\begin{aligned}\rho(Y_{u_1}^{w(1)}\ldots Y_{u_q}^{w(q)})+\rho(Y_w^i) &= (|u_1|-1)+\ldots+|u_q|-1)+(|w|-1)\\ &= |u|-q+(q-1)\\ &= \rho(Y_u^j),\end{aligned}$$

and thus

$$\Delta(Y_u^i)\in\sum_{p+q=n}\mathcal{L}_q\otimes\mathcal{L}_q.$$

Given $y\in H_p$, $y'\in H_{p'}$, $z\in H_q, z'\in H_{q'}$, we have that $(x\otimes y)(x'\otimes y')\in H_{p+p'}\otimes H_{q+q'}$. Since Δ is a multiplicative homomorphism, we conclude that if P is an arbitrary interval, i.e., a product of terms of the form Y_u^i, and $\rho(P)=m$, then

$$\Delta(P)\in\sum_{p''+q''=m}\mathcal{L}_{p''}\otimes\mathcal{L}_{q''}$$

and thus $\mathcal{L}$ is a graded and connected bialgebra.

The following antipode formula of Schmitt may be regarded as a transcription of (5.2) (see [4], §11.1 and [10], Th. 4.1).

$$S(Y_v^i)=\sum_{k\geq 0}\sum_{0_v=\sigma_0\prec\sigma_1\prec\ldots\prec\sigma_k=1_j}(-1)^k\prod_{i=1}^{k}[\sigma_{\iota-1},\sigma_i]. \tag{6.2}$$

On the other hand the subspaces $\mathcal{L}_n$ are finite dimensional, and thus from above S is invertible and S^{-1} is the antipode for $\mathcal{L}^\tau$.

We may express S^{-1} in terms of S. Let $\mathcal{L}^{op}$ denote the opposite algebra of $\mathcal{L}$. Given a word $u=u_1\cdots u_n\in[N]^*$, we let $u^*=u_n\ldots u_1$. Since $\mathcal{L}$ is freely generated by the Y_u^j, the inclusion mapping

$$\mathfrak{s}:Y_u^i\mapsto Y_{u^*}^i\in\mathcal{L}^{op}$$

extends to an algebra isomorphism $\mathfrak{s}:\mathcal{L}\to\mathcal{L}^{op}$. This may be regarded as an antiisomoprhism $\sigma:\mathcal{L}\to\mathcal{L}$ satisfying

$$\mathfrak{s}(Y_{u_1}^{i_1}\cdots Y_{u_n}^{i_n})=Y_{u_n^*}^{i_n}\cdots Y_{u_1^*}^{i_1}. \tag{6.3}$$

Lemma 4. $S^{-1}=\mathfrak{s}S\mathfrak{s}$.

Proof. We have that

$$\begin{aligned}\sum\mathfrak{s}S\mathfrak{s}((Y_u^i)_{(2)})(Y_u^i)_{(1)} &= \sum\mathfrak{s}S(Y_{(v_n\cdots v_1)})Y_{u_1}^{v(1)}\ldots Y_{u_n}^{v(n)}\\ &= \mathfrak{s}(\sum Y_{u_n^*}^{v(n)}\ldots Y_{u_1^*}^{v(1)}S(Y_{u_n\cdots u_1}))\\ &= \mathfrak{s}(\sum(Y_{u^*}^i)_{(1)}S(Y_{u^*})_{(2)})\\ &= \mathfrak{s}(\varepsilon(Y_{u^*}^i)1)=\delta_{u^*}^i=\delta_u^i.\end{aligned} \tag{6.4}$$

We have a related anti-isomorphism $\mathfrak{t} : \mathcal{L} \to \mathcal{L}$ determined by the identity mapping

$$\mathfrak{t} : Y^i_u \mapsto Y^i_u \in \mathcal{L}^{op},$$

or equivalently,

$$\mathfrak{t}(Y^{i(1)}_{u_1} \cdots Y^{i(n)}_{u_n}) = Y^{i(n)}_{u_n} \cdots Y^{i(1)}_{u_1}$$

7. The reduced tree formula for the reverse antipode

Given a reduced colored tree $T \in \mathcal{R}^i_u$, all of the vertices x of T are totally ordered by the (decreasing) depth first ordering. Each vertex x determines a corresponding generator $Y(x) = Y^j_v$, where j is the color of x, and $v = v(1) \dots v(k)$ are the colors (in order) of its children. We define

$$\Lambda(T) = \prod_{x \in T}^{\ggg} Y(x) = Y(x_r) \cdots Y(x_1), \tag{7.1}$$

where $x_1 \lll \dots \lll x_r$ are the non-leaf vertices of T.

Given an ordered set of n reduced trees $T_1, \dots , T_n$. We define $c_x(T_1, \dots , T_n)$ to be the tree obtained by introducing a new colored root x with color i_0, and joining each of the roots x_j (with color i_j) of T_j to x_0. We let $\mathbf{v}(T)$ denote the number of non-leaf vertices in a reduced tree T.

It is evident that with the exception of the unique one layer tree $T^i_u \in \mathcal{R}^i_u$, every tree $T \in \mathcal{R}^i_u$ has a unique representation of the form $T = c_x(T_1, \dots , T_n)$.

Lemma 5. *Suppose that we are given an ordered n-tuple of trees $(T_1, \dots , T_n) \in \mathcal{R}$ $(n \geq 2)$, and that the root x_j of T_j has color i_j. Then we have*

$$\Lambda(c_x(T_1, \dots , T_n)) = Y^{i_0}_{i_1 \dots i_n} \Lambda(T_n) \dots \Lambda(T_1) \tag{7.2}$$

and $\mathbf{v}(\Lambda(c_x(T_1, \dots , T_n)) = \sum \rho(T_j) + 1$.

Proof. Let us suppose that the non-degenerate vertices of T_k are given by $x_{k,1} \lll \dots \lll x_{k,p_k}$. The new root x_0 is non-degenerate in the tree $T = c_x(T_1, \dots , T_n)$ and it is maximal in the $\lll$ ordering. It follows that

$$\begin{aligned} \Lambda(T) &= Y(x)Y(x_{n,p_k})Y(x_{n,p_k-1}) \cdots Y(x_{1,p_1}) \cdots Y(x_{1,1}) \\ &= Y^{i_0}_{i_1 \dots i_n} \Lambda(T_n) \dots \Lambda(T_1). \end{aligned}$$

The second relation is immediate. ∎

Theorem 6. *The antipode S of $\mathcal{L}$ is determined by*

$$S^{-1}(Y^i_u) = \sum_{T \in \mathcal{R}^i_u} (-1)^{\mathbf{v}(T)} \Lambda(T). \tag{7.3}$$

Proof. Since S^{-1} satisfies $S^{-1}(1) = 1$, and it is an antihomomorphism, this relation indeed determines S on $\mathcal{L}$. We use the recursive characterization (5.3). We have that if $u = jk$, $\mathcal{R}^i_u$ contains only the tree $T = T^i_{jk}$ and $\Lambda(T^i_{jk}) = Y^i_{jk}$. Since $Y^i_{jk} \in \mathcal{L}_1$, we have from (**??**),

$$S^{-1}(Y^i_{jk}) = -Y^i_{jk} = (-1)^{\mathbf{v}(T)} \Lambda(T),$$

which coincides with the right side of (7.3).

Let us suppose that the formula is true for $|u| \leq n-1$. We have from (??) that if $u = u(1)\dots u(n+1)$,

$$S^{-1}(Y_u^i) = -Y_u^i - \sum (Y_u^i)_{(2)} S^{-1}((Y_u^i)_{(1)})$$

(recall that $\varepsilon(Y_u^i) = 0$). On the other hand from (6.1)

$$\Delta(Y_v^i) = \sum_{q=1}^{p} \sum_{\pi \in \mathcal{I}_q(p)} \sum_{w \in [n]^q} Y_{v_1}^{w(1)} \dots Y_{v_q}^{w(q)} \otimes Y_w^i$$

and thus letting $\pi = \{B_1, \dots, B_q\}$, and $u_k = u|B_k$,

$$\begin{aligned}
S^{-1}(Y_u^i) &= -Y_u^i - \sum_{q,\pi,w} Y_w^i S^{-1}(Y_{u_1}^{w(1)} \cdots Y_{u_q}^{w(q)}) \\
&= -Y_u^i - \sum_{q,\pi,w} Y_u^i S^{-1}(Y_{u_q}^{w(q)}) \cdots S^{-1}(Y_{u_1}^{w(1)}) \\
&= -Y_u^i - \sum_{q,\pi,w} \sum_{T_k \in \mathcal{R}_{u_k}^{w(k)}} (-1)^{\sum \mathbf{v}(T_k)} Y_w^i \Lambda(T_q) \cdots \Lambda(T_1) \\
&= -Y_u^i + \sum_{q,\pi,w} \sum_{T_k \in \mathcal{R}_{u_k}^{w(k)}} (-1)^{\mathbf{v}(c_x(T_1,\dots,T_q))} \Lambda(c_x(T_1, \dots, T_q)) \\
&= -Y_u^i + \sum_{T \in \mathcal{R}_u^i \setminus \{T_u^i\}} (-1)^{\mathbf{v}(T)} \Lambda(T)
\end{aligned}$$

where we recall that the one layer tree T_u^i is not assembled from non-trivial reduced subtrees. On the other hand we have that $\Lambda(T_u^i) = Y_u^i$ and $\mathbf{v}(T_u^i) = 1$, hence

$$S^{-1}(Y_u^i) = \sum_{T \in \mathcal{R}_u^i} (-1)^{\mathbf{v}(T)} \Lambda(T)$$

8. The breadth first formula for the antipode

For the breadth first formula for the antipode, we restrict our attention to layered trees. Given such a tree T, each non-degenerate vertex of $x \in T(\gamma)$ determines a generator $Y(x) = Y_v^j$ where j is the color of x and v is the coloring of its children. We define

$$\Omega(T) = \prod_{x \in T_{nd}}^{\ll} Y(x) = Y(x_1) \dots Y(x_r), \tag{8.1}$$

where $x_1 \ll \dots \ll x_r$ are the non-degenerate vertices in T. We have from (6.2) that

$$S(Y_v^i) = \sum_{T \in \mathcal{LT}_v^i} (-1)^{\ell(T)} \Omega(T)$$

where $\mathcal{LT}_v^i$ is the set of all layered trees with root colored i and leaves colored by v, and $\ell(T)$ is the number of layers in T.

It would be tempting to attempt to use (8.1) to obtain a formula with reduced trees by simply contracting the edges issuing from degenerate vertices. In the commutative situation considered by Haiman and Schmitt, one could associate a reduced tree $R = \rho(T)$ with any layered tree T by contracting the edges issuing from degenerate vertices. In this commutative context $\Omega(\rho(T)) = \Omega(T)$ and thus in the formula for the antipode one may collect all the terms with the same contracted tree into a multiple of $\Omega(\rho(T))$. The coefficient is a sum of positive and negative 1's, and using a combinatorial argument they showed that all of the non-reduced trees cancel.

In our situation, arbitrary contractions can disturb the $x \ll y$ ordering on the non-degenerate leaves, and thus one need not have that $\Omega(\rho(T)) = \Omega(T)$. This can be seen in the third tree of the diagram below, which was disordered by an "improper" contraction. One must therefore use only contractions which are $\ll$ *order preserving*. In our reduction we will also modify the contraction so that the tree remains layered.

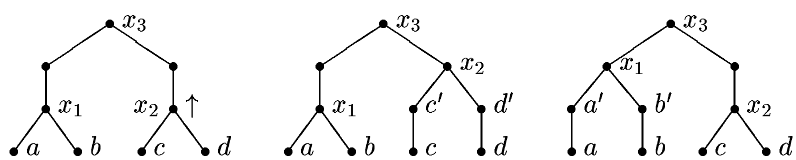
Let us suppose that T is a layered tree and that x is a non-degenerate vertex in T. We say that T is *order contractible at x* if

a) its parent x' is degenerate
b) there does not exist a non-degenerate vertex to the *right* of x
c) there does not exist a non-degenerate vertex to the *left* of x'.

If x is a vertex in the k-th row which satisfies these conditons, the *order contraction* $\kappa(T) = \kappa_x(T)$ is the layered tree obtained in the following manner:

1) move x to the position of its parent in the $(k-1)$-st row,
2) attach each child y of x by a single line to a degenerate vertex x' in the k-th row,
3) leave all other vertices and edges alone.
4) if there are no other non-degenerate vertices in the k-th row, delete it.

Conditions a)-c) guarantee that the $\ll$ ordering on the non-degenerate vertices is preserved. Thus the contraction on the non-degenerate vertex x_2 in the first tree below is allowed. On the other hand contracting on the vertex x_1 would transpose the $\ll$ ordering for the two non-degenerate vertices x_1 and x_2.

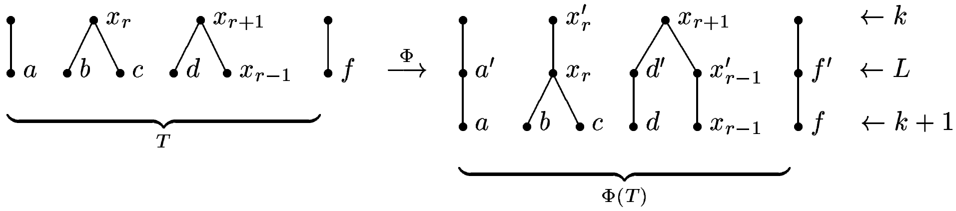


Since we will not consider general contractions, we will simply use the terms *contractible* and *contractions* for the corresponding order preserving notions.

Given $j \in [N]$ and $v \in [N]^*$, we let $\mathcal{T}_v^j$ be the set of all layered trees with root colored by j, and leaves colored by v. We let $\mathcal{ET}_v^j$ be the simple trees in $\mathcal{T}_v^j$. Given a tree $T \in \mathcal{LT}_v^j$ we define the *canonical expansion* $\Phi(T) \in \mathcal{T}_v^j$ as follows. If T is simple we let $\Phi(T) = T$. If T is not simple, let x_r be the first non-simple non-degenerate vertex in the $\ll$ ordering, and let us suppose that it is on the k-th level. We introduce a new level L between the k-th and $(k+1)$-st levels in the following manner.

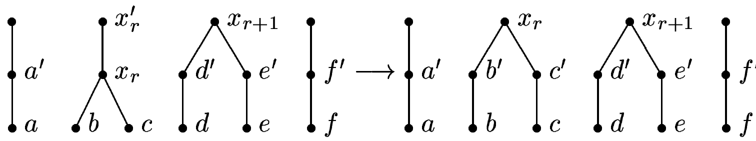
a) We move x_r down to the level L and we connect it to a new degenerate vertex x_r' on the k-th level, and to the children of x_r on the $(k+1)$-st level,
b) If y is a vertex on the $(k+1)$-st that is not a child of x_r, we connect it by a single edge to a new degenerate vertex y' on the level L, which we then connect to the parent of y on the k-th level.

We define $\Phi(T)$ to be the new tree.



It should be noted that since there are no non-degenerate vertices to the left of x_r, this operation will not affect the $\ll$ ordering on the non-degenerate vertices. It also preserves the ordering $\lll$ on the non-degenerate vertices, as is evident from the above diagram.

We have that x_r is a contractible vertex in $\Phi(T)$ because all the other vertices on the new level are degenerate, and there are no non-degenerate vertices to the left of x_r'. If one contracts on this vertex, the new level will contain only degenerate vertices, and thus will itself be deleted (see the primed row in the right tree below). In this manner we see that if we contract $\Phi(T)$ at the vertex x_r, we recover T. This is illustrated in the following diagram, in which $e = x_{r-1}$ and $e' = x'_{r-1}$.



Let us suppose that $x_1 \ll \ldots \ll x_p$ are the non-degenerate vertices of a tree T. Turning to the breadth first ordering, there is a unique sequence of indices $n_1 < \ldots < n_q$ with

$$\ldots \lll x_{n_1} \ggg x_{n_1+1} \ggg x_{n_1+2} \ggg \ldots \ggg x_{n_2} \lll x_{n_2+1} \ldots$$

We call a maximal sequence of the form $x_{n_{k-1}+1} \ggg x_{n_{k-1}+2} \ggg \ldots \ggg x_{n_k}$ an *irreducible string* and we say that x_{n_k} is its *right end.*

Lemma 7. *Suppose that T is an arbitrary tree with sequence .*

(i) Any contractible vertex in T is a right end of an irreducible string.

(ii) If $T = E$ is simple then all of its right ends are contractible.

(iii) If one has $y \ll x$ in T and both y and x are contractible, then after a contraction at x, y will still be contractible.

Proof. (i)If $x = x_j$ is a contractible vertex in T on level k. Then in particular its parent x_j' is degenerate and there are no vertices to the left of its parent. Since every level is assumed to have a non-degenerate vertex, x_{j+1} must lie on the $(k-1)$-st row of E to the right of the parent x_j' of x_{j+1}. It follows that $x_j \ll x_{j+1}$, and thus $x_j = x_{n_h}$ for some h.

(ii) Let us suppose that $y_h = x_{n_h}$ is a non-degenerate vertex on the k-the level. There are no non-degenerate vertices to the right of x_{n_h} on its level since the level is simple. But $x_{n_h} \lll x_{n_h+1}$ shows that the parent y_h' of y_h lies to the left of x_{n_h+1}, there are no non-degenerate vertices to the left of y_h' on the $(k-1)$-st level. Thus there are no obstructions to the liftability of y_h.

(iii) If x is simple, then the contraction at x will simply raise the level of each vertex y with $y \ll x$. If x is not simple, then x is the only non-degenerate vertex that is affected. Since y is assumed contractible, there won't be any vertices to the left of it on its level. On the other hand if y is on level $k+1$, then by the same assumption, x must lie to the right of the parent y'. This will still be be the case when one contracts x to a higher level. ∎

For each simple tree E we let $\mathcal{T}_E$ be all the trees $T \in \mathcal{T}_v^j$ with $E = \Phi^n(T)$ for some n. It is evident that if T has n vertices and k levels, then $E = \Phi^{n-k}(T)$ is a simple tree, and reversing the expansions as above, T can be obtained by a particular sequence of contactions of E. More precisely, let $y_1 \ll \ldots \ll y_q$ be the right vertices of T (or equivalently of E). We have that there is a subsequence $y_{m_1} \ll \ldots \ll y_{m_p}$ with

$$T = \kappa_{y_{m_1}} \ldots \kappa_{y_{m_p}}(E)$$

Conversely given any such sequence, the subsequent right vertices remain contractible as one proceeds, and we get a corresponding tree T. The tree T uniquely determines the sequence $y_{m_1} \ll \ldots \ll y_{m_p}$ since the latter are by definition the non-simple nondegenerate vertices of T in their given $\ll$ order.

We let $\mathcal{NST}_v^j$ be the set of all non-order contractible simple trees in $\mathcal{T}_v^j$.

Theorem 8. *The antipode in is given by*

$$S(Y_v^i) = \sum_{E \in \mathcal{NST}_v^i} (-1)^{\ell(E)} \Omega(E)$$

where $\ell(T)$ is the number of layers in T.

Proof. It is evident that

$$\mathcal{T}_v^j = \sqcup \{\mathcal{T}_E : E \in \mathcal{ST}_v^j\}$$

and that for any $T \in \mathcal{T}_E$ we have that $\Omega(T) = \Omega(E)$. Thus it suffices to show that if E has contractions, then

$$\sum_{T \in \mathcal{T}_E} (-1)^{\ell(T)} \Omega(T) = \Omega(E) \sum_{T \in \mathcal{T}_E} (-1)^{\ell(T)} = 0.$$

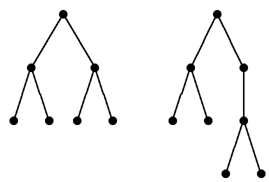
From our earlier discussion, $\mathcal{T}_E$ is in one-to-one correspondence with the sequences $y_{m_1} \ll \ldots \ll y_{m_p}$ drawn from the q right vertices in E, or equivalently subsets drawn from $1, \ldots, q$. If the simple tree E has n non-degenerate vertices and thus n levels, the tree $T(m_1, \ldots, m_p)$ has $n-p$ levels. There will be $\binom{q}{p}$ such sequence and thus

$$\sum_{T \in \mathcal{T}_E} (-1)^{\ell(T)} = (-1)^{n-q} \sum_p \binom{q}{p} (-1)^{q-p} = (-1)^{n-q}(1-1)^q = 0,$$

and we have proved the desired result. ∎

9. The duality between depth first and breadth first orderings

There is a natural one-to-one correspondence between the reduced trees $\mathcal{R}_u^j$and the non-contractible simple layered trees. $\mathcal{NST}_v^j$. On the one hand we have the mapping $\rho : \mathcal{NST}_u^j \to \mathcal{R}_u^j$ in which one contracts all the edges emanating from degenerate vertices. We define $\eta : \mathcal{R}_u^j \to \mathcal{NST}_u^j$ by forcing simplicity via appropriate expansions. This is illustrated in the following diagram in which a reduced tree is transformed into a non-contractible simple tree (one can make the second tree layered by adding degenerate branches).



In this correspondence the breath order and depth first orders are interchanged. We will not pursue these notions at this time.

10. Formal Power Series

Let us suppose that we are given a non-commutative unital algebra A and non-commuting variables $z_1, \ldots, z_N$. Given a word $w = w(1)\cdots w(p) \in [N]^*$, we let $z_w = z_{w(1)}\cdots z_{w(p)}$, and $z_e = 1$. A multiple non-commutative power series with N non-commuting variables $z_1, \ldots, z_N$ and non-commuting constants has the form $F(z) = (F^1(z_1, \ldots z_n), \ldots, F^N(z_1, \ldots, z_N))$ where

$$F^j(z) = F^j(z_1, \ldots, z_N) = \sum f^j_w z_w,$$

and the "constants" f^j_w lie in A. We assume that variables commute with constants. The latter enables us to multiply power series since in particular,

$$(az_v)(bz_w) = abz_{vw}.$$

Let us begin by computing the effect of *substitution* on power series. We do not use the term "composition" since there does not seem to be a meaningful interpretation along those lines. Given a single power series of N variables

$$F(z) = f_e + \sum f_j z_j + \sum f_{jk} z_j z_k + \cdots$$

and an N-tuple of power series without constant terms

$$G^i(z) = \sum g^i_j z_j + \sum f_{jk} z_j z_k + \cdots$$

we may substitute $G^j(z)$ for z_j in the expression or F. We will denote the resulting power series by $H(z) = (F \circ G)(z)$. Rather than doing this explicitly, it is more instructive to compute the coefficient h_w of z_w. A typical summand of $h_{w_1 \ldots w_p}$ is obtained by taking an ordered interval partition of $\pi = \{C_1, \ldots, C_p\}$,of $(w_1, \ldots, w_p)$. Given that $C_1 = \{1, \ldots n_1\}$, we have a corresponding expression

$$g^{i_1}_{w_1 \ldots w_{p_1}} z_{w_1} z_{w_2} \ldots z_{w_{p_1}} = g_{w|C_1} z^{w|C_1}.$$

and we have corresponding expressions for each C_j. The relevant summand of $h_{w_1 \ldots w_p}$is given by

$$f_{j_1 \cdots j_q} g^{j_1}_{w|C_1} \cdots g^{j_q}_{w|C_q}.$$

We conclude that

$$h_{w_1 \ldots w_p} = \sum_q \sum_{\pi = (C_k) \in \mathcal{I}_q(N)} f_{j_1 \cdots j_q} g^{j_1}_{w|C_1} \cdots g^{j_q}_{w|C_q}.$$

More generally we may substitute G into an M-tuple $F(z) = (F^1(z), \ldots, F^n(z))$, obtaining $H = F \circ G$, where

$$h^i_{w_1 \ldots w_p} = \sum_q \sum_{\pi=(C_k) \in \mathcal{I}_q(N)} f_{i_1 \ldots i_q} g^{i_1}_{w|C_1} \cdots g^{i_q}_{w|C_q}.$$

Given an algebra A, we let $G^{dif}_N(A)$ denote the set of power series $F = F(z)$ with

$$F^j(z) = z_j + \sum_{|u| \geq 2} f^j_u z_u, \quad (f^j_u \in A)$$

i.e., without constant terms and with $f^i_j = \delta^j_i$. Substitution of G into F provides us with a non-associative product $(F, G) \mapsto F \circ G$ on $\mathcal{G}^{dif}_N(A)$. From above,

$$(F \circ G)^i_u = z_i + \sum f^i_w g^{w(1)}_{u|C_1} \cdots g^{w(s)}_{u|C_s} \tag{10.1}$$

where if $p = |u|$, we sum over q-interval colored partitions $\pi = ((C_1, \ldots, C_q), w)$ $(1 \leq q \leq p)$ of $[p]$.

Each generator $Y^i_u \in \mathcal{L}_N$ $(|u| > 1)$ may be used to select a corresponding coefficent f^i_u in a power series $F(z)$. To be more precise, we define a linear mapping

$$\theta(Y^i_u) : G^{dif}_N(A) \to A$$

by letting $\theta(Y^i_u)(F) = f^i_u$. Since $\mathcal{L}_N$ is the free algebra on these generators, we extend this to the basis elements $Y^{i_1}_{u_1} \cdots Y^{i_q}_{u_q}$ by letting

$$\theta(Y^{i_1}_{u_1} \cdots Y^{i_q}_{u_q}) : \mathcal{G}^{dif}_N(A) \to A : F \mapsto f^{i_1}_{u_1} \cdots f^{i_q}_{u_q}$$

Extending linearly, we have a corresponding homomorhism

$$\theta : \mathcal{L}_N \to Lin(\mathcal{G}^{dif}_N(A), A)$$

and thus a bilinear mapping

$$\langle , \rangle : \mathcal{L}_N \times \mathcal{G}^{dif}_N \to A : (a, F) \mapsto \theta(a)(F).$$

From our definitions we have that

$$\langle ab, f \rangle = \langle a, f \rangle \langle b, f \rangle = m_A \langle a \otimes b, f \otimes f \rangle.$$

We let $\tau : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ be the flip defined by $\tau(a \otimes b) = b \otimes a$. Returning to the substitution formula (10.1). we have

$$\begin{aligned} \langle Y^i_u, F \circ G \rangle &= m_A \sum \langle Y^i_w \otimes Y^{w(1)}_{u|C_1} \cdots Y^{w(s)}_{u|C_s}, F \otimes G \rangle \\ &= m_A \langle \Delta^\tau(Y^i_w), F \otimes G \rangle. \end{aligned}$$

Theorem 9. *Given $F \in G^{dif}_N(A)$, the left substitutional inverse of F is given by the power series $G(z)$, where $g^j_v = \langle S^{-1}(Y^j_v), f \rangle$.The right substitutional inverse of F is given by H, where $h^j_v = \langle \tilde{S}(Y^j_v), f \rangle$, where $\tilde{S} = \mathfrak{t} S \mathfrak{t}$.*

Proof. : Let m_A denote the multiplication in A. Defining G as above, we have that

$$\begin{aligned}
\langle Y_u^j, G \circ F \rangle &= m_A \langle \Delta^\tau (Y_u^j), G \otimes F \rangle \\
&= m_A (\sum \langle Y_w^i \otimes Y_{u|C_1}^{w(1)} \cdots Y_{u|C_s}^{w(s)}, G \otimes F \rangle) \\
&= \sum \langle S^{-1}(Y_w^j), F \rangle \langle Y_{u|C_1}^{w(1)} \cdots Y_{u|C_s}^{w(s)}, F \rangle \\
&= m_A (\sum \langle S^{-1}((Y_u^j)_{(2)})(Y_u^j)_{(1)}, F \rangle \\
&= \langle \sum S((Y_u^j)_{(2)})(Y_u^j)_{(1)}, F \rangle \\
&= \langle \varepsilon(Y_u^j) 1, F \rangle \\
&= \delta_u^j 1,
\end{aligned}$$

and thus F is the left substitutional inverse of G.

On the other hand, if H is defined as above, then using the fact that $\mathfrak{t}S$ is an algebraic homomorphism and that $\mathfrak{t}(Y_w^i) = Y_w^i$,

$$\begin{aligned}
\langle Y_u^j, F \circ H \rangle &= m_A \langle \Delta^\tau (Y_u^j), F \otimes H \rangle \\
&= m_A (\sum \langle Y_w^i \otimes Y_{u|C_1}^{w(1)} \cdots Y_{u|C_s}^{w(s)}, F \otimes H \rangle) \\
&= m_A (\sum \langle Y_w^i \otimes (\mathfrak{t}S)(Y_{u|C_1}^{w(1)}) \cdots (\mathfrak{t}S)(Y_{u|C_s}^{w(s)})), F \otimes F \rangle) \\
&= m_A (\sum \langle Y_w^i \otimes (\mathfrak{t}S)(Y_{u|C_1}^{w(1)}) \cdots Y_{u|C_s}^{w(s)}), F \otimes F \rangle \\
&= \sum \langle Y_w^j (\mathfrak{t}S)(Y_{u|C_1}^{w(1)} \cdots Y_{u|C_s}^{w(s)})), F \rangle \\
&= \langle \mathfrak{t}(\sum S(Y_{u|C_1}^{w(1)} \cdots Y_{u|C_s}^{w(s)}) Y_w^j), F \rangle \\
&= \langle \mathfrak{t}(\sum S(Y_u^j)_{(1)} (Y_u^j)_{(2)}), F \rangle \\
&= \langle \varepsilon(Y_u^j) 1, F \rangle \\
&= \delta_u^j 1
\end{aligned}$$

and H is the right substitutional inverse of F.

Corollary 10. *If the number of variables N is greater than 1, then the left and right substitutional inverses of a power series are generally distinct.*

Proof. It suffices to show that

$$\sigma \circ S \circ \sigma(Y_{1234}^1) \neq \tilde{\sigma} \circ S \circ \tilde{\sigma}(Y_{1234}^1)$$

In the following calculation we have used boldface subscripts to indicate corresponding terms that equal. In each sum the bracketed terms cancel (these correspond to order preserving contractions).The sums are over the set of colors 1,2,3,4.

$$\begin{aligned}
S(Y^1_{1234}) &= -Y^1_{1234} + \sum Y^k_{12}Y^1_{k34} + \sum Y^k_{23}Y^1_{1k4} \\
&\quad + \sum Y^k_{34}Y^1_{12k} + \left[\sum Y^k_{12}Y^\ell_{34}Y^1_{k\ell}\right] - \sum Y^k_{34}Y^\ell_{12}Y^1_{\ell k} - \left[\sum Y^k_{12}Y^\ell_{34}Y^1_{k\ell}\right] \\
&\quad - \sum Y^k_{12}Y^\ell_{k3}Y^1_{\ell 4} - \sum Y^k_{23}Y^\ell_{1k}Y^1_{\ell 4} - \sum Y^k_{23}Y^\ell_{k4}Y^1_{1\ell} - \sum Y^k_{34}Y^\ell_{2k}Y^1_{1\ell} \\
\sigma \circ S \circ \sigma(Y^1_{1234}) &= -Y^1_{1234} + \ \mathbf{a}\sum Y^1_{k34}Y^k_{12} + \ \mathbf{b}\sum Y^1_{1k4}Y^k_{23} \\
&\quad + \ \mathbf{c}\sum Y^1_{12k}Y^k_{34} + \left[\sum Y^1_{kl}Y^\ell_{34}Y^k_{12}\right] - \sum Y^1_{\ell k}Y^\ell_{12}Y^k_{34} - \left[\sum Y^1_{k\ell}Y^\ell_{34}Y^k_{12}\right] \\
&\quad - \ \mathbf{d}\sum Y^1_{\ell 4}Y^\ell_{k3}Y^k_{12} - \ \mathbf{e}\sum Y^1_{\ell 4}Y^\ell_{1k}Y^k_{23} - \ \mathbf{f}\sum Y^1_{1\ell}Y^\ell_{k4}Y^k_{23} - \ \mathbf{g}\sum Y^1_{1\ell}Y^\ell_{2k}Y^k_{34} \\
\tilde\sigma(Y^1_{1234}) &= Y^1_{4321} \\
S\tilde\sigma(Y^1_{1234}) &= S(Y^1_{4321}) = -Y^1_{4321} + \sum Y^k_{43}Y^1_{k21} + \sum Y^k_{32}Y^1_{4k1} \\
&\quad + \sum Y^k_{21}Y^1_{43k} + \left[\sum Y^k_{43}Y^\ell_{21}Y^1_{k\ell}\right] - \sum Y^k_{21}Y^\ell_{43}Y^1_{\ell k} - \left[\sum Y^k_{43}Y^\ell_{21}Y^1_{k\ell}\right] \\
&\quad - \sum Y^k_{43}Y^\ell_{k2}Y^1_{\ell 1} - \sum Y^k_{32}Y^\ell_{4k}Y^1_{\ell 1} - \sum Y^k_{32}Y^\ell_{k1}Y^1_{4\ell} - \sum Y^k_{21}Y^\ell_{3k}Y^1_{4\ell} \\
\tilde\sigma S\tilde\sigma(Y^1_{1234}) &= -Y^1_{1234} + \ \mathbf{c}\sum Y^1_{12k}Y^k_{34} + \ \mathbf{b}\sum Y^1_{1k4}Y^k_{23} \\
&\quad + \ \mathbf{a}\sum Y^1_{k34}Y^k_{12} + \left[\sum Y^1_{\ell k}Y^\ell_{12}Y^k_{34}\right] - \sum Y^1_{k\ell}Y^\ell_{34}Y^k_{12} - \left[\sum Y^1_{\ell k}Y^\ell_{12}Y^k_{34}\right] \\
&\quad - \ \mathbf{g}\sum Y^1_{1\ell}Y^\ell_{2k}Y^k_{34} - \ \mathbf{f}\sum Y^1_{1\ell}Y^\ell_{k4}Y^k_{23} - \ \mathbf{e}\sum Y^1_{\ell 4}Y^\ell_{1k}Y^k_{23} - \ \mathbf{d}\sum Y^1_{\ell 4}Y^\ell_{k3}Y^k_{12}
\end{aligned}$$

It follows that

$$\sigma \circ S \circ \sigma(Y^1_{1234}) - \tilde\sigma S\tilde\sigma(Y^1_{1234}) = -\sum Y^1_{\ell k}Y^\ell_{12}Y^k_{34} + \sum Y^1_{k\ell}Y^\ell_{34}Y^k_{12}.$$

∎

In particular, one can check that if a and b do not commute, the substitional left and right inverses of the two-variable polynomial function

$$\begin{aligned} u &= x + ax^2 + by^2 \\ v &= y \end{aligned}$$

do not agree in the fourth order terms.

As in pointed out in BFK, the paradoxical fact that power series with non-commuting coefficients and variables need not be associative can be related to a "free analogue" of the Hopf algebra which uses a non-coassociative coproduct

$$\Delta_* : \mathcal{L} \to \mathcal{L}{*}\mathcal{L}.$$

We will not pursue this idea in this paper.

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