Try to do this yourself before you look at the solutions. I may have messed up the algebra here or there, if you find any mistakes email me at hwadhar@math.ucla.edu and let me know.

Good luck.

1. Part a: This is the region in the first quadrant between $\frac{1}{\sqrt{2}} \leq \rho \leq 1$. The fact that we are in the first quadrant gives us that $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi/2$. I included a diagram below, make sure you can come up with this yourself.

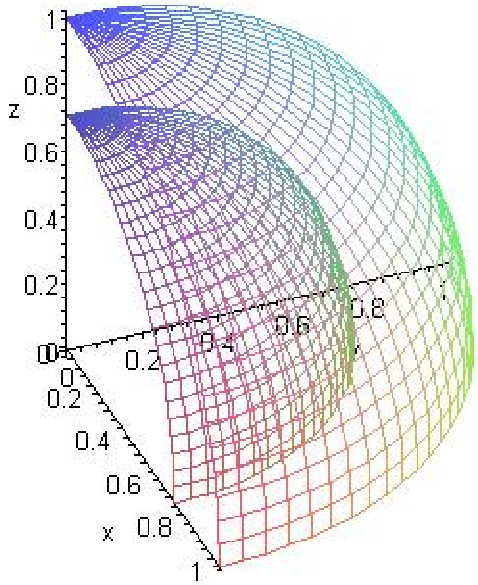

Figure 1: The volume we are interested in is in between the two spheres shown above.

Part b: To translate between spherical and cartesian coordinates we use the following transformation:

$$\begin{aligned} x &= \rho \cos\theta \sin\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\phi \end{aligned}$$

Then we can write the integral as follows:

$$\begin{aligned} \int\int\int_B \sqrt{x^2+y^2+z^2}dV &= \int_{\frac{1}{\sqrt{2}}}^{1}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sqrt{x^2+y^2+z^2}|J|dV \\ &= \int_{\frac{1}{\sqrt{2}}}^{1}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \rho(\rho^2\sin\phi)d\phi d\theta d\rho \\ &= \int_{\frac{1}{\sqrt{2}}}^{1}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \rho^3\sin\phi d\phi d\theta d\rho \end{aligned}$$

Everyone should be able to go through and evaluate this integral. The answer I get is $\frac{3\pi}{32}$.

2. Let $(x, y) = T(u, v)$ be define by $x = u^2 - v^2$, $y = 2uv$.

Part a: The region R in the uv plane is shown below.

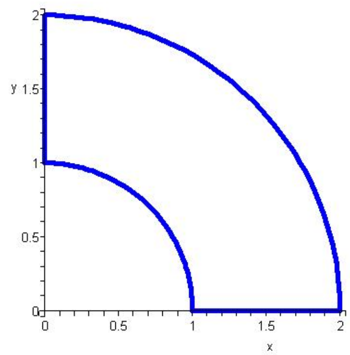

Figure 2: R is given by the region inside the blue curves above.

We know that the $E = T(R)$, so lets see what $T(R)$ is. To do this we look at each of the four curves (the two quarter circles and the two portions that lie along the x and y axis respectively) and see where they are mapped to by the transformation.
First we see that $x^2 + y^2 = (u^2 + v^2)^2$, so the radii of the two circles in R are squared when we transform to E in xy space. Check how to do the lines yourself, but I get that E looks like this:

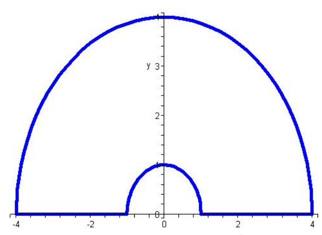

Figure 3: $E = T(R)$ is given by the region inside the blue curves above.

Now we need to use this transformation to evaluate the integral

$$\int\int_E \frac{1}{\sqrt{x^2 + y^2}} dA$$

First we will calculate the Jacobian of this transformation:

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} \\ &= 4(u^2 + v^2) \end{aligned}$$

Then we can rewrite the integral as

$$
\begin{aligned}
\int\int_E \frac{1}{\sqrt{x^2+y^2}}dA &= \int\int_R \frac{1}{u^2+v^2}|J|dA \\
&= \int\int_R \frac{4}{u^2+v^2}(u^2+v^2)dA \\
&= \int\int_R 4dA \\
&= 4A(R) \\
&= 3\pi
\end{aligned}
$$

3.

(a) Sketch the vector field $\vec{F}(x, y) = [y, -x]$.

(b) Determine $\int_C \vec{F} \cdot d\vec{r}$ where C is the unit circle at the origin traversed in a counterclockwise direction from the positive x axis to the negative x axis.

In general this is what you need to do to compute a line integral:

(a) Sketch your curve and some points on the vector field.

(b) Parameterize your curve.

(c) Compute the time derivative of your parametrization.

(d) Rewrite integral as follows:

$$\int_C \vec{F}(x, y) d\ r = \int_{t_0}^{t_1} \vec{F}(x(t), y(t)) \cdot \frac{d\vec{r}}{dt} dt$$

(e) Do the integral (some people forgot to do this on the last exam)

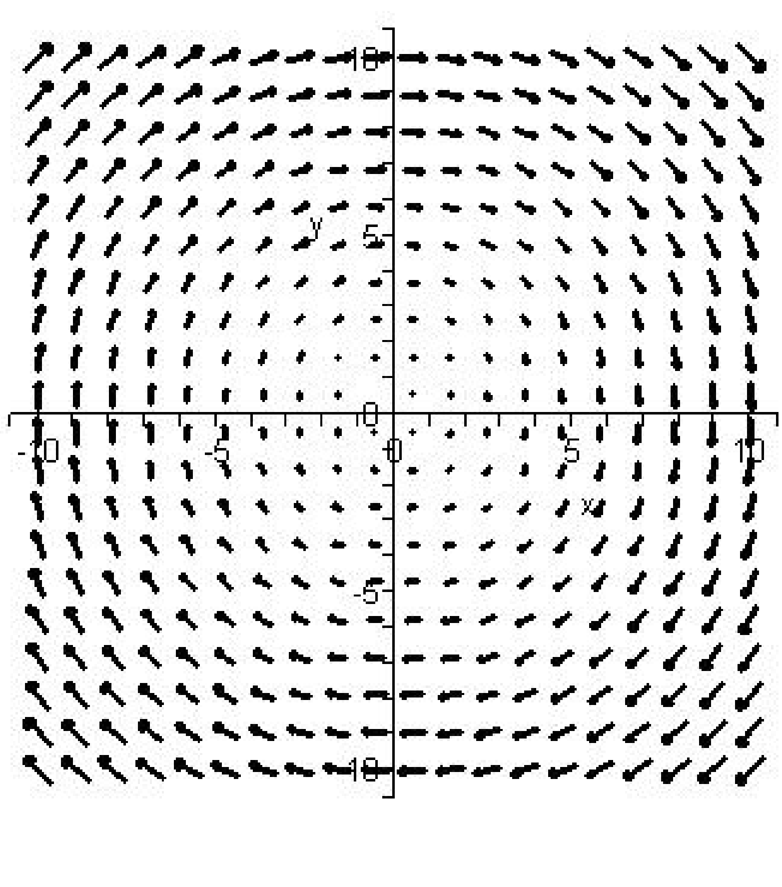

Figure 4: Here is the plot of the vector field.

Part b: The problem asks us to consider the portion of the unit circle in the upper half plane oriented counterclockwise. Knowing how to parameterize (semi-, quarter-) circles, straight lines and helices is something you should absolutely know. Looking above at the field plot you should be able to tell right away that the sign of this line integral over this path is negative.

The parametrization of the curve is given by:

$$\vec{r}(t) = [x(t), y(t)] = [\cos t, \sin t]$$

for $0 \leq t \leq \pi$. Be sure to check that this curve (or one on your exam) goes in the direction you want it to go.

The derivative of our parametrization is

$$\frac{d\vec{r}}{dt} = [-\sin t, \cos t]$$

Now we can compute the integral as follows

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_{t_0}^{t_1} \vec{F}(x(t), y(t)) \cdot \frac{d\vec{r}}{dt} dt \\
&= \int_0^{\pi} [\sin t, -\cos t] \cdot [-\sin t, \cos t] dt \\
&= -\int_0^{\pi} (\sin^2 t + \cos^2 t) dt \\
&= -\int_0^{\pi} (1) dt \\
&= -\pi
\end{aligned}$$

4. Determine the following line integrals:

(a) $\int_\Gamma y dx + x^2 dy$, where Γ is given by $x = t$ and $y = t^2$ from $-1 \le t \le 2$. Include a sketch of the curve.

(b) $\int_T y dx + x^2 dy$, where T is the triangle determined by the points (0, 0), (1, 2) and (-1, 3) traversed in a counterclockwise direction. Include a sketch.

Part a: This is the parametrization of a parabola since $x = t$ and $y = t^2 = x^2$ going from left to right.

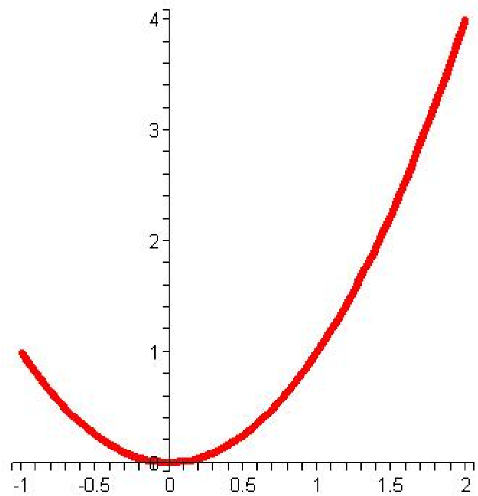

Figure 5: Here is the curve Γ, the arrow points to the right.

Now let us rewrite the integral as follows:

$$\int_\Gamma y dx + x^2 dy = \int_\Gamma [y, x^2] \cdot [dx, dy] = \int_\Gamma [y, x^2] \cdot d\vec{r}$$

Now we know that $\vec{r}(t) = [t, t^2]$, which is then tells us that $\frac{d\vec{r}}{dt} = [1, 2t]$. Then we can rewrite the integral above like this:

$$\int_\Gamma [y, x^2] \cdot d\vec{r} \quad = \quad \int_{-1}^{2} [t^2, t^2] \cdot [1, 2t] dt$$

Then doing the integral out you should get that the value of the integral is $\frac{21}{2}$.

Part b: In this problem we will have to compute three line integrals for each line segment along the triangle.

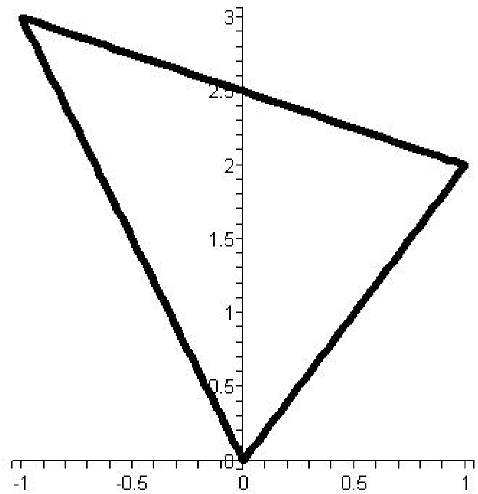

Figure 6: Here is the curve T, the arrows point so that the curve goes clockwise.

Let C_1 connect (1, 2) to (0, 0), and C_2 connect (0, 0) to (-1, 3), and C_3 connect (-1, 3) to (1,2). Then the parameterizations are

$$\begin{aligned} \vec{r}_1(t) &= [1-t, 2-2t] & \frac{d\vec{r}_1}{dt} &= [-1,-2] \\ \vec{r}_2(t) &= [-t, 3t] & \frac{d\vec{r}_2}{dt} &= [-1,3] \\ \vec{r}_3(t) &= [2t-1, 3-t] & \frac{d\vec{r}_3}{dt} &= [2,-1] \end{aligned}$$

Then the integral becomes

$$\begin{aligned} \int_T [y,x^2]\cdot d\vec{r} &= \int_{C_1} [y,x^2]\cdot d\vec{r}_1 + \int_{C_2} [y,x^2]\cdot d\vec{r}_2 + \int_{C_3} [y,x^2]\cdot d\vec{r}_3 \\ &= \int_{C_1} [2-2t,(1-t)^2]\cdot[-1,-2]dt + \int_{C_2} [3t,(-t)^2]\cdot[-1,3]dt + \int_{C_3} [3-t,(2t-1)^2]\cdot[2,-1]dt \\ &= \frac{5}{2} \end{aligned}$$

Check my algebra there is a very good chance I made a few mistakes.