Printed Name__________________________Signature____________________UID______

Math 131a S04 Final Examination June 15, 2004 $\rightarrow$ Answer 8 questions $\leftarrow$

1. Prove that any subset of $\mathbb{N}$ is either finite or countably infinite.

2. Prove that if $x_n \to L$ and $x_n > 0$ and $L > 0$, then there exists a $c > 0$ such that $x_n \geq c$ for all n.

3. Prove that if $f(x)$ is continuous on $[a, b]$ and $f(a) < p < f(b)$, then there exists a $c \in (a, b)$ such that $f(c) = p$.

4. Define a function $f(x)$ on $[0,1]$ by

$$f(x) = \begin{cases} 1 & x\text{rational} \\ 0 & x \text{ irrational} \end{cases}$$

Is $f(x)$ Riemann integrable? Carefully prove your assertion.

5. Prove that if f is a continuous function on $[a,b]$ and $f(x) \geq 0$ for all x, then $\int_a^b f(x)dx = 0$ implies that $f(x) = 0$ for all x. Hint: Suppose that $f(p) = \varepsilon > 0$. Show that $f \geq g$ where g is equal to ε on a small interval $[p-\delta, p+\delta]$, and is zero off of that interval.

6. Prove that if $f(x)$ is continuous on $[a,b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

7. (a) Prove that if $f(x)$ is defined on (a, b) and differentiable at $c \in (a, b)$, then f is continuous at c.

   (b) Is the converse of a) true? Prove your assertion.

8. Give examples, and include careful proofs of your assertions

   (a) A sequence of continuous functions $f_n(x)$ on $[0, 1]$ which converges to a continuous function $f(x)$, but for which $\int_0^1 f_n(x)dx$ does not converge to $\int_0^1 f(x)dx$.

   (b) A sequence of continuous functions $f_n(x)$ on $[0, 1]$ which converges to a non-continuous function $f(x)$ on $[0, 1]$.

(c) A continuous function on $\mathbb{R}$ which is not uniformly continuous on $\mathbb{R}$.

9. (a) Define

    i. (M, ρ) is a metric space.

    ii. The sequence $x_n \in M$ converges to $x \in M$.

    iii. The sequence $x_n \in M$ is Cauchy.

    iv. M is a complete metric space.

   (b) Prove that if x_n is a sequence which converges to x, then x_n is Cauchy.

   (c) Use a specific example to show the converse is false.