Handout #5
Assigment 4:
p. 86:1,2,3,6,7,9,10
p. 94: 1,2,3,4,7,8

Here are some more things that you should know:

- f is continuous at a point c if $x_n \to c$ implies that $f(x_n) \to f(c)$.
- You should be able to prove that if f and g are continuous at c, then so are $f+g$, fg, and you should be able to state and prove similar results for $f \circ g$ and f/g [the easiest approach is to use the limit laws for sequences].
- How to state and prove the fact that f is continuous at c if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x \in [a,b]$, $|x-c| < \delta$ implies that $|f(x) - f(c)| < \varepsilon$. You should be able to use this criterion to prove that various functions are continuous.
- The difference between "f is continuous on a set S" and "f is uniformly continuous on a set S".
- You should be able to give examples of continuous functions which are not uniformly continuous, and prove your assertion.
- You should be able to prove that if f is a continuous function on $[a,b]$, then

  a) f is bounded on $[a,b]$, i.e., there exists an M such that $|f(x)| \leq M$ for all $x \in [a,b]$.

  b) f assumes its maximum and minimum values on $[a,b]$.

  c) If $f(a) < r < f(b)$, then there is a $c \in [a,b]$ such that $f(c) = r$.

  d) f is uniformly continuous on $[a,b]$.

  You should be able to explain why we had to assume f was defined on a closed bounded interval in a), b), and d).

Here is a result that we proved in class which is not in the book:

Theorem: Suppose that x_n is an arbitrary sequence of real numbers. Then it has a monotone subsequence.

Proof: Let S be the "locations n with a view", i.e.,

$$S = \{n \in \mathbb{N} : m > n \text{ implies } x_m < x_n\}.$$

Case 1: Suppose that S is infinite. We may let $S = \{n(1), n(2), \ldots\}$ where $n(1) < n(2) < \ldots$ (this follows from a simple induction). We have that $x_{n(1)} > x_{n(2)} > x_{n(3)} > \ldots$.

Case 2: Suppose that S is finite. Then let $N = \max S$ and let $n(1) = N + 1$. Since $n(1) \notin S$, the set $\{m > n(1) : x_m \geq x_{n(1)}\}$ is non-empty. Let $n(2)$ be the first integer in that sense. Again since $n(2) > n(1) > \max S$, $n(2) \notin S$ and thus the set $\{m > n(2) : x_m \geq x_{n(2)}\}$ is non-empty. Continuing in this fashion, we get a subsequence $x_{n(1)} \leq x_{n(2)} \leq \dots$ QED