Handout 3a

Some standard uncountable sets.

Given a set S, we let $\mathcal{P}(S)$ denote the set of all subsets A of S. For example

$$\mathcal{P}(\{1.2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.$$

If S is a finite set with n elements, then $\mathcal{P}(S)$ has 2^n elements.

Given two sets S and T, we let $\mathcal{F}(S,T)$ denote the set of all functions $f : S \to T$. If S has m elements and T has n elements then $\mathcal{F}(S,T)$ has m^n elements. Because for each $t \in T$ you have m possible choices for $s \in T$ with $f(s) = t$.

Theorem: $\mathcal{F}(\mathbb{N}, \{0,1\})$ is not countable.

Proof: A function $a : \mathbb{N} \to \{0,1\}$ may be identified with a sequence $(a_1, a_2, \dots)$ where $a_k = a(k)$ is 0 or 1. If $\mathcal{F}(\mathbb{N}, \{0,1\})$ is countably infinite, then let $F : \mathbb{N} \approx \mathcal{F}(\mathbb{N}, \{0,1\})$. For example suppose that

$$\begin{aligned} F(1) &= (\mathbf{1}, 0, 1, 1, \dots) \\ F(2) &= (0, \mathbf{1}, 1, 0, \dots) \\ F(3) &= (0, 0, \mathbf{0}, 1, \dots) \\ &\dots \end{aligned}$$

To get a contradiction we will display a sequence which is not in this list. Simply define $b_n = 1 - F(n)_n$. In the above example,

$$\begin{aligned} b_1 &= 1 - \mathbf{1} = 0 \\ b_2 &= 1 - \mathbf{1} = 0 \\ b_3 &= 1 - \mathbf{0} = 1. \\ &\dots \end{aligned}$$

(we are simply flipping the diagonal term. Then $b \neq F(1)$, $b \neq F(2), b \neq F(3), \dots$ because $b_1 = 1 - F(1)_1 \neq F(1)_1$, and in general $b \neq F(n)$ since $b_n = 1 - F(n)_n \neq F(n)_n$. QED.

Corollary: $\mathcal{P}(\mathbb{N})$ is not countable.

Proof: We have a 1-1 correspondence $F : \mathcal{F}(\mathbb{N}, \{0,1\}) \approx \mathcal{P}(\mathbb{N})$ given by $F(f) = f^{-1}(0)$ (f is completely determined by where it is 0).

Corollary: $\mathbb{R}$ is not countable.

Proof: Define a map $\theta : \mathcal{F}(\mathbb{N}, \{0,1\}) \to \mathbb{R}$ by letting

$$\theta(((a_1, a_2, \dots)) = \frac{a_1}{10} + \frac{a_2}{10^2} + \dots$$

— this is just the decimal expansion $.a_1a_2a_3\dots$ θ is one-to-one. To see this suppose that

$$a = (a_1, a_2, \dots) \neq b = (b_1, b_2, \dots).$$

We let n be the first integer with $b_n \neq a_n$. Let us assume that $a_n = 0$ and $b_n = 1$ so that

$$\begin{aligned} a &= (a_1, \dots, a_{n-1}, 0, a_{n+1}, \dots). \\ b &= (a_1, \dots, a_{n-1}, 1, b_{n+1}, \dots) \end{aligned}$$

We have that

$$\begin{aligned}
\theta(a) &= \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{a_{n+1}}{10^{n+1}} + \ldots \\
&\leq \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{1}{10^{n+1}} + \frac{1}{10^{n+2}} \ldots \\
&= \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{1}{10^n}[\frac{1}{10} + \frac{1}{10^2} \ldots] \\
&= \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{1}{10^n} \cdot \frac{1}{9} \\
&< \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{1}{10^n} \\
&\leq \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{1}{10^n} + \frac{b_{n+1}}{10^{n+1}} + \ldots \\
&= \frac{b_1}{10} + \frac{b_2}{10^2} + \ldots + \frac{b_{n-1}}{10^{n-1}} + \frac{b_n}{10^n} + \frac{b_{n+1}}{10^{n+1}} + \ldots \\
&= \theta(b).
\end{aligned}$$

We let S be the image of $\mathcal{F}(\mathbb{N}, \{0,1\})$ (this is all the numbers which can be represented by a decimal expansion with only 0's and 1's and no integer part). We have

$$\mathcal{F}(\mathbb{N}, \{0,1\}) \approx S \subseteq \mathbb{R}.$$

If $\mathbb{R}$ were countable, it would follow that S and therefore $\mathcal{F}(\mathbb{N}, \{0,1\})$ is countable (why?), which is a contradiction of the theorem.