Math 131a Handout #3

**We will assume two fundamental properties of** $\mathbb{N}$ :

(**N1**) Every non-empty subset $S$ of $\mathbb{N}$ has a least element.

(**N2**) The Fundamental Theorem of Arithmetic: every number $n \in \mathbb{N}$ has a unique factorization

$$n = 2^{a_1} 3^{a_2} \dots$$

where $0 \le a_k \in \mathbb{N} \bigcup \{0\}$ .

**Theorem 0.1.** *Suppose that $S$ is an infinite subset of a countable set $T$. Then $S$ is countably infinite (i.e., $S \approx \mathbb{N}$).*

*Proof.* First assume that $T = \mathbb{N}$. We define a function $f : \mathbb{N} \to T$ by induction. From **(N1)** we may let $f(1) = \min S$. Let us suppose that we have defined $f(n-1)$ (where $n > 1$). Since $S$ is assumed infinite, $S \backslash \{f(1), \dots, f(n-1)\}$ is non-empty, and we may use **(N1)** to define

$$f(n) = \min S \backslash \{f(1), \dots, f(n-1)\} .$$

It is evident that

$$f(1) < f(2) < \dots$$

and in particular $f$ **is 1-1**.

**To see that** $f$ **is onto** we have to show that if $p \in S$, then there is an $n$ such that $f(n) = p$. First observe that for all $n \in \mathbb{N}$, $n \le f(n)$. To see this note that $1 \le f(1)$ since 1 is the least element in all of $\mathbb{N}$. Suppose that we know that $n \le f(n)$. Then $n \le f(n) < f(n+1)$ implies that $n+1 \le f(n+1)$ (note that $f(n+1)$ is a "whole" number). Thus induction gives the general result $\forall n, n \le f(n)$.

Given $p \in S$, let $A = \{n \in \mathbb{N} : p \le f(n)\}$ . This is non-empty since $p \le f(p)$. Let $n_0 = \min A$. If $n_0 = 1$, then

$$f(1) = \min S \le p.$$

and thus $f(1) = p$. If $n_0 > 1$, then

$$f(1) < \dots < f(n_0 - 1) < p \le f(n_0),$$

and thus $p$ is in $S \backslash \{f(1), \dots, f(n_0 - 1)\}$ . It follows that

$$f(n_0) = \min S \backslash \{f(1), \dots, f(n_0 - 1)\} \le p,$$

and thus $f(n_0) = p$..

For the general case, by assumption $T \approx \mathbb{N}$, i.e., there is a bijection $g : T \to \mathbb{N}$. Then $g(S)$ is an infinite subset of $\mathbb{N}$, and by our prevous argument $g(S) \approx \mathbb{N}$. Since $S \approx g(S)$, $S \approx \mathbb{N}$, i.e., $S$ is countably infinite. □

**Theorem 0.2.** *Suppose that $T$ is a countable set $T$ and $f : T \to U$ is onto. Then $U$ is countable.*

*Proof.* Since $f$ is onto, we have that for each $u \in U$, the set $T_u = \{t : f(t) = u\}$ is non-empty. For each $u \in U$, we choose an element $t_u \in T_u$. We define $g : U \to T$ by $g(u) = t_u$. From this definition $f(g(u)) = u$. It follows that $g : U \to T$ is one-to-one since if $g(u_1) = g(u_2)$, then $f(g(u_1)) = f(g(u_2))$ and thus $u_1 = u_2$. It is evident that $g$ is a one-to-one correspondence of $U$ onto the set $g(U)$, i.e., $U \approx g(U) \subseteq T$. Since $g(U)$ is infinite, we conclude from the previous result that $g(U) \approx \mathbb{N}$, and thus $U \approx \mathbb{N}$. □

**Theorem 0.3** (The principle of induction)**.** *Suppose that one has a series of statements $P(1)$, $P(2), \ldots$. Then if $P(1)$ is true, and $P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$, then $P(n)$ is true for all $n$.*

*Proof.* Let us suppose that this is false. Then there exists an $n \in \mathbb{N}$ such that $P(n)$ is false**. Thus the set

$$S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$$

is non-empty. Using (**N1**), we may let $n_0 = \min S$. Since $P(1)$ is assumed true, $n_0 > 1$. From the definition of $n_0$, $P(n_0 - 1)$ is true, and $P(n_0)$ is false, contradicting the fact that for all $n$, $P(n) \Rightarrow P(n+1)$*. ☐

*This illustrates the law of logic $[\sim (Q \Rightarrow R)] \Leftrightarrow [Q \text{ and } \sim R]$.

** This illustrates the law of logic $[\sim (\forall x \in X) P(x)] \Leftrightarrow [(\exists x \in X) \sim P(x)]$.

**Completeness axiom for $\mathbb{R}$ : Any set which is bounded above has a least upper bound.**

Using letters: if you have a subset $S \subseteq \mathbb{R}$ such that $S \leq b$ for some $b \in \mathbb{R}$ (i.e., $s \leq b$ for all $s \in S$), then $S$ has a **least upper bound** $b_0$ (i.e., $S \leq b_0$ and if $S \leq b$ then $b_0 \leq b$).

**Theorem 0.4.** *$\mathbb{N}$ does not have an upper bound.*

*Proof.* Suppose that $\mathbb{N}$ has an upper bound. Then using the completeness principle, we may let $b_0 = \sup \mathbb{N}$ be the least upper bound for $\mathbb{N}$. We have that $b_0 - 1 < b_0$ implies that $b_0 - 1$ is not an upper bound for $\mathbb{N}$ i.e., $\mathbb{N} \not\leq b_0 - 1$ and there is an $n \in \mathbb{N}$ with $b_0 - 1 < n$. But then $b_0 < n + 1 \in \mathbb{N}$, contradicting the fact that $b_0$ is an **upper bound** for $S$. QED

**Corollary 0.5.** *For any $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.*

*Proof.* Since $\mathbb{N}$ is not bounded above, there is an $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. It follows that $\frac{1}{n} < \varepsilon$. QED

**Corollary 0.6.** *If $a > 0$ and $b > 0$, there is an $n \in \mathbb{N}$ such that $na > b$.*

*Proof.* You prove this!

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**Assignment 2**

(1) p. 54: 1,2
(2) Given complete proofs that
  a) $\lim_{n\to\infty} \frac{n}{n+1} = 1$
  b) $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$
  c) $\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$ (use the sandwich principle and a geometrical picture)
(3) What can be said if $a_n$ is a convergent sequence in $\mathbb{N}$?
(4) Consider the set

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \ldots$$

For which numbers $a$ is there a subsequence converging to $a$?
(5) a) Show that if $0 < a < 2$, then $a < \sqrt{2a} < 2$.

b) Prove that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converges.
c) Find the limit of the sequence in b).

(6) p. 51: 7, 11
(7) p. 54: 5, 6.