Math 131A/1 Handout #2

Some pointers on the logical conventions of mathematics.

It is probably hardest to get used to the way mathematicians use the word "implies" or the symbol $\Rightarrow$. The idea is that you want to know if the implication is true or not just on the basis of whether the ingredients are true or false **without any more thinking**.

- Any true statement $\Rightarrow$ any true statement (e.g., $1 + 1 = 2 \Rightarrow$ there are infinitely many primes $\Rightarrow$ Fermat's last conjecture)
- Any false statment $\Rightarrow$ any true statement (because, in particular, you want to be able to say that $1 = 2 \Rightarrow 1 \times 0 = 2 \times 0$ is a correct "deduction").
- Any false statement $\Rightarrow$ any false statement (because, in particular, you want to be able to say that 1=2 $\Rightarrow 1+1 = 2+1$ is a correct "deduction").
- The following is **false:** truth $\Rightarrow$ false

1. The most common logical **errors** made by beginners:
   - They think that "or" is exclusive: thus although they know that $\leq$ means "less than or equal to" they think it is "wrong" to write $3 \leq 3$ because they know that "actually, $3 = 3$"
   - They think that you cannot prove $P \Rightarrow Q$ if you already know that P is false.
   - They think that if $P \Rightarrow Q$ is true, then Q is true.
   - When asked to prove $P \Rightarrow Q$ they instead prove $Q \Rightarrow P$.
   - They get = (equality - say of numbers) mixed up with $\Leftrightarrow$ (logical equivalence, used for propositions).
2. Some *correct* illustrations of logic:
   - The proposition "$6 < 7$ or $4 < 5$" is true.
   - The proposition "$1 = 2 \Rightarrow 0 \times 1 = 0 \times 2$" is true.
   - The proposition "For any real number x, $x+2 = 3 \Rightarrow x(x+2) = x3$" is true.
   - The proposition "For any real number x, $x(x+2) = x3 \Rightarrow x+2 = 3$" is false.
3. It is important to be able to take the negations of statements (in order to prove things by contradiction). Here is the general scheme (don't worry about the last two at this point)

$$\begin{array}{rcl}
\tilde{}(P \text{ or } Q) & \Leftrightarrow & (\tilde{}P) \text{ and } (\tilde{}Q) \\
\tilde{}(P \text{ and } Q) & \Leftrightarrow & (\tilde{}P) \text{ or } (\tilde{}Q) \\
\tilde{}(P & \Rightarrow & Q) \Leftrightarrow (P \text{ and } \tilde{}Q) \\
\tilde{}((\forall x)P(x)) & \Leftrightarrow & (\exists x)(\tilde{}P(x)) \\
\tilde{}((\exists x)P(x)) & \Leftrightarrow & (\forall x)(\tilde{}P(x))
\end{array}$$

Here is an important tautology for statements with two variables:

$$(\exists x)(\forall y)P(x, y) \Rightarrow (\forall y)(\exists x)P(x, y)$$

You will have to become adept at using absolute values. Recall that

Def.:

$$|x| = \left\{ \begin{array}{cc} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{array} \right\} = (x^2)^{1/2}$$

You should be able to prove:

- $\forall x \in \mathbb{R}, |x| = 0$ if and only if $x = 0$
- $\forall x \in \mathbb{R},\ x \leq |x|$
- $\forall x, c \in \mathbb{R},\ |x| \leq c$ if and only if $-c \leq x \leq c$ if and only if $(x \leq c$ and $-x \leq c)$
- $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$
- $\forall x, y \in \mathbb{R}, |xy| = |x|\,|y|$
- $||x| - |y|| \leq |x - y|$