Handout 8
Reminder: The final is being held in a different room (see announcement on the webpage).
Some further definitions, theorems, examples, and their proofs you should know for the final examination.

- Suppose that f is bounded on $[a, b]$. Given a partition
$$P : a = x_0 < \ldots < x_n = b,$$
the lower and upper Riemann sums are defined by
$$\begin{aligned} L_P(f) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ U_P(f) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \end{aligned}$$
where $m_i = \inf \{f(x) : x_{i-1} \le x \le x_i\}$ and $M_i = \sup \{f(x) : x_{i-1} \le x \le x_i\}$. If $P \le Q$ (i.e., Q contains the points of P) then
$$L_P(f) \le L_Q(f) \le U_Q(f) \le U_P(f).$$
It follows that for any partitions P and Q, $L_P(f) \le U_Q(f)$ (**why**?) and we may define
$$\begin{aligned} \underline{\int_a^b} f(x)dx &= \sup \{L_P(f) : P \text{ a partition of } [a, b]\} \\ \overline{\int_a^b} f(x)dx &= \inf \{U_Q(f) : Q \text{ a partition of [a,b]}\}. \end{aligned}$$
We say that f is Riemann integrable if $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$. Equivalently, for any $\varepsilon > 0$, there exists a partition P with
$$U_P(f) - L_P(f) < \varepsilon.$$

- Suppose that f is a continuous function on $[a, b]$. Then f is Riemann integrable. In fact if one is given a sequence of partitions
$$P_k : a = x_0^k < \ldots < x_{r(k)}^k = b$$
with
$$mesh(P_k) = \max \left\{ \left| x_i^k - x_{i-1}^k \right| : i = 1, \ldots, r_k \right\} \to 0,$$
and $x_i^{k*} \in [x_{i-1}^k, x_{i-1}^k]$, we let
$$S_{P_k}(f) = \sum_{i=1}^{r(k)} f(x_i^{k*})(x_i^k - x_{i-1}^k),$$
then
$$\int_a^b f(x)dx = \lim_{k \to \infty} S_{P_k}(f).$$

- An example of a discontinuous function that is nevertheless Riemann integrable.

- The proofs that for continuous functions f and g,
$$\begin{aligned} \int_a^b (f+g)(x)dx &= \int_a^b f(x)dx + \int_a^b g(x)dx \\ \int_a^b cf(x)dx &= c \int_a^b f(x)dx \\ f \le g &\Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx \\ \left| \int_a^b f(x)dx \right| &\le \int_a^b |f(x)|\, dx \end{aligned}$$

- Integral Mean Value Theorem: if f is continuous on $[a,b]$, then there exists a $c \in [a,b]$ such that
$$\frac{\int_a^b f(x)dx}{b-a} = f(c).$$
- If $a < c < b$, then $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
- Suppose that f is defined on (a,b) and $c \in (a,b)$. We say that f is differentiable at c if
$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$
exists. If that is the case we denote these limits by $f'(c)$.
- Prove: If f is differentiable at $c \in (a,b)$, then it is continuous at c.
- The proof that if f and g are differentiable at c, then
$$\begin{aligned} (f+g)'(c) &= f'(c) + g'(c) \\ (fg)'(c) &= f'(c)g(c) \end{aligned}$$
and if $g'(c) \neq 0$ then
$$(1/g)'(c) = \frac{g'(c)}{g(c)^2},$$
If g is differntiable at $f(c)$, then one has the chain rule
$$(g \circ f)'(c) = g'(f(c))f'(c).$$
- If f is differentiable at c and f assumes a maximal or minimal value on $[a,b]$ at c. then $f'(c) = 0$.
- Rolle's Theorem: Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b) = 0$, then for some $c \in (a,b)$, $f'(c) = 0$.
- Differentiable mean value theorem (MVT): Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then for some $c \in (a,b)$,
$$\frac{f(b) - f(a)}{b-a} = f'(c)$$
- The simple corollaries of the MVT.
- **Statments and proofs of the two fundamental theorems of calculus.**
- The definition of convergence and uniform convergence for functions f_n and f on a set $S \subseteq \mathbb{R}$..
- You should be able to verify uniform convergence. Sometimes it is useful to use some calculus for this purpose. On the other hand you should be able to use the criterion that f_n does **not** converge uniformly to f if and only if there exists an $\varepsilon > 0$ and sequences x_n, x'_n such that $|x_n - x'_n| \to 0$ but $|f(x_n) - f(x'_n)| \geq \varepsilon$ for all n .
- The proof that if $f_n : S \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly, then f is continuous.
- The proof that $f_n : [a,b] \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly, then $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.
- Counterexamples showing why you need uniform convergence in the above two results.
- The proof that if $f_n : [a,b] \to \mathbb{R}$ are differentiable and $f_n(x) \to f(x)$, and $f'_n(x) \to g(x)$ uniformly, then $f' = g$.
- An example showing that you can have f'_n continuous and $f_n \to f$ uniformly, but f is not differentiable at some point.

- The definition of a normed space $(V, \| \|)$, and the examples $(\mathbb{R}^n, \| \|_p)$, $p = 1, 2, \infty$, as well as $C([a, b]), \| \|_\infty$. You should be able to prove these are all complete normed spaces.
- The definition of a metric space (M, d), and why we may regard a subset of a normed space as a metric space.
- The definition of convergence and Cauchy for sequences in a metric space. The defintion of when a metric space is complete, and examples of non-complete metric spaces.
- The statement and proof of the contraction mapping theorem for complete metric spaces.