Handout #7

- Definition:$\lim_{x\to c} f(x) = L$ if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

- We have that $\lim_{x\to c} f(x) = L$ if and only if for all sequences x_n with $x_n \to c$ and $x_n \neq c$, $f(x_n) \to L$.

  Proof: Say that $\lim_{x\to c} f(x) = L$. Then suppose $x_n \to c$ with $x_n \neq c$. Given $\varepsilon > 0$ choose $\delta > 0$ as above. Then choose n_0 such that $n \geq n_0$ implies $|x_n - c| < \delta$. It follows that $|f(x_n) - L| < \varepsilon$. Conversely suppose that $\lim_{x\to c} f(x) \neq L$. Then

$$\exists \varepsilon > 0 : \forall \delta > 0, \exists x : 0 < |x - c| < \delta \text{ and } |f(x) - L| \geq \varepsilon.$$

  Using this with $\delta = 1/n$, where $n \in \mathbb{N}$ choose x_n so that $0 < |x_n - c| < \delta$ and $|f(x) - L| \geq \varepsilon$. Then $x_n \to c$, $x_n \neq c$, but $f(x_n) \nrightarrow L$.

- Definition: f is continuous at c if $\lim_{x\to c} f(x) = f(c)$. Equivalently,

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, 0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

  or since $f(c) = L$,

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

  Another equivalent statement:

$$\text{If } x_n \to c, \text{ then } f(x_n) \to f(c).$$

- You should be able to prove that if f is a continuous function on $[a, b]$, then

  a) f is bounded on $[a, b]$, i.e., there exists an M such that $|f(x)| \leq M$ for all $x \in [a, b]$.

     Proof: Suppose that f is not bounded on $[a, b]$. Choose x_n such that $|f(x_n)| \geq n$. From the Balzano Weierestrass theorem there exists a convergent susequence $x_{n_k} \to c$. It follows that $f(x_{n_k}) \to f(c)$, and thus the sequence $f(x_{n_k})$ is bounded, contradicting the fact that $|f(x_{n_k})| \geq n_k$.

  b) f assumes its maximum and minimum values on $[a, b]$. Let $M = \sup \{f(x) : x \in [a, b]\}$. Then we may choose a sequence x_n such that $f(x_n) \to M$ (why?). Let $x_{n_k} \to c$ (how do you know you can do this?). We have that $f(x_{n_k}) \to f(c)$ and thus $f(c) = M$. Same argument for m.

c) If $f(a) < p < f(b)$, then there is a $c \in [a, b]$ such that $f(c) = p$.

Proof: Let $S = \{x : f(x) < p\}$ and let $c = \sup S$ (how do you know this exists?). Then there is a sequence $x_n \in S$ such that $x_n \to c$. It follows that $f(x_n) \to f(c)$. Since $f(x_n) < p$, we have $f(c) \leq p$ (why?) We claim that $f(c) \geq p$. If not, $f(c) < p$. We get a contradiction as follows. Let $\varepsilon = p - f(c)$. Choose $\delta > 0$ such that $|x - c| < \delta$ implies that $|f(x) - f(c)| < \varepsilon$. Then $|x - c| < \delta$ implies that

$$f(x) < f(c) + \varepsilon = p$$

and thus $x \in S$. In particular letting $x = c + \delta/2$, we see that $c + \delta/2 \in S$ and thus $c + \delta/2 \leq c$, a contradiction.

Here is a result we've used before:

Theorem: Suppose that x_n is an arbitrary sequence of real numbers. Then it has a monotone subsequence.

Proof: Let S be the "locations n with a view", i.e.,

$$S = \{n \in \mathbb{N} : m > n \text{ implies } x_m < x_n\}.$$

Case 1: Suppose that S is infinite. We may let $S = \{n(1), n(2), \ldots\}$ where $n(1) < n(2) < \ldots$ (this follows from a simple induction). We have that $x_{n(1)} > x_{n(2)} > x_{n(3)} > \ldots$.

Case 2: Suppose that S is finite. Then let $N = \max S$ and let $n(1) = N + 1$. Since $n(1) \notin S$, the set $\{m > n(1) : x_m \geq x_{n(1)}\}$ is non-empty. Let $n(2)$ be the first integer in that sense. Again since $n(2) > n(1) > \max S$, $n(2) \notin S$ and thus the set $\{m > n(2) : x_m \geq x_{n(2)}\}$ is non-empty. Continuing in this fashion, we get a subsequence $x_{n(1)} \leq x_{n(2)} \leq \ldots$ QED