Math 131a Handout #6

Our **completeness axiom** : If S is a non-empty subset of $\mathbb{R}$, and S is bounded above (i.e., $S \leq b$ for some b), then S has a least upper bound $b_0 = \sup S$. (You fomulate the corresponding result for non-empty sets that are bounded below).

Here are theorems about sequences and their limts that you should be able to prove (including the relevant definitions):

- If x_n is a convergent sequence, then it must be bounded.

  Proof: Suppose that $x_n \to L$. Choose n_0 such that $n \geq n_0 \Rightarrow |x_n - L| < 1$. Then $|x_n| - |L| \leq |x_n - L| < 1$ implies that $|x_n| < |L| + 1$. Let $M = \max\{|x_1|, \ldots, |x_{n_0-1}|, |L| + 1\}$. We have that for *all* n $|x_n| \leq M$.

- If $x_n \to L$ and $x_n \neq 0$ and $L \neq 0$, then there is a constant $c > 0$ such that $|x_n| \geq c$ for all n.

  Proof: Suppose first that $x_n > 0$ and $L > 0$. Choose n_0 such that $n \geq n_0 \Rightarrow |x_n - L| < L/2$. Then $L - x_n \leq |x_n - L| < L/2$ implies that $x_n > L - L/2 = L/2$. Let $c = \min\{x_1, x_2, \ldots, x_{n_0-1}, L/2\}$. It follows that $|x_n| \geq c$ for all n. For the general case note that $|x_n| \to |L|$ and use the positive result.

- If $x_n \to L$ and for all n, $x_n \geq 0$, then $L \geq 0$.

  Proof: Suppose that $L < 0$. Then let $\varepsilon = -L$. We may choose n_0 such that $|x_n - L| < \varepsilon$. Then $x_{n_0} - L < \varepsilon \Rightarrow x_{n_0} < L + \varepsilon = 0$, contradicting $x_{n_0} \geq 0$.

- The usual limit theorems (such as $x_n \to L$ and $y_n \to M$ implies $x_n + y_n \to L + M$).

- If x_n is an increasing sequence, and $x_n \leq b$, then $x_n \to b_0 = \sup\{x_n\}$. (You should be able to state and prove the corresponding result for decreasing sequences).

  Proof: Given $\varepsilon > 0$, we have that $b_0 - \varepsilon < b_0$ implies that $b_0 - \varepsilon$ is not an upper bound for $\{x_n\}$, hence there exists an n_0 with $b_0 - \varepsilon < x_{n_0}$. It follows that if $n \geq n_0$, then $b_0 - \varepsilon < x_{n_0} \leq x_n \leq b_0$. and thus $|x_n - b_0| < \varepsilon$.

- If $\emptyset \neq S \subseteq \mathbb{R}$ and $b_0 = \sup S$, then there is a sequence $x_n \in S$ such that $x_n \to b_0$. (You should be able to state and prove the corresponding result for the infimum).

  Proof: Given $n \in \mathbb{N}$, $b_0 - 1/n$ is not an upper bound for S, hence we may choose an $x_n \in S$ such that $b_0 - 1/n < x_n \leq b_0$. It follows that$|x_n - b_0| < 1/n$, .and thus $x_n \to b_0$.

- If x_n is a convergent sequence, then it must be Cauchy.

  Proof: Say that $x_n \to L$. Given $\varepsilon > 0$, choose n_0 such that $n \geq n_0$ implies that $|x_n - L| < \varepsilon/2$. Then $m, n \geq n_0$ implies that $|x_n - x_m| = |(x_m - L) + (L - x_m)| \leq |x_m - L| + |x_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

- If x_n is a Cauchy sequence, then it must be bounded.

  Proof: Given $\varepsilon > 0$, choose n_0 such that $m, n \geq n_0$ implies that $|x_m - x_n| < 1$. Then in particular $m \geq n_0 \Rightarrow |x_m - x_{n_0}| < 1$ and thus $|x_m| < 1 + |x_{n_0}|$. Let $M = \max\{|x_1|, \ldots, |x_{n_0-1}|, |x_{n_0}| + 1\}$. We have that $x_n \leq M$ for all n.

- If $x_n \to L$, then for any subsequence x_{n_k}, $x_{n_k} \to L$.

- Every sequence has a monotonic subsequence.[See handout 3a]

- If x_n is a Cauchy sequence and a subsequence $x_{n_k} \to L$, then $x_n \to L$.

  Proof Given $\varepsilon > 0$ choose n_0 such that $m, n \geq n_0$ implies that $|x_m - x_n| < \varepsilon/2$. Choose k_0 such that $k \geq k_0$ implies that $|x_{n_k} - L| < \varepsilon/2$. Then since $n_k \geq k$, we may also assume that $n_{k_0} \geq n_0$. Then if $n \geq n_0$, $\left|x_n - x_{n_{k_0}}\right| < \varepsilon/2$. It follows that if $n \geq n_0$ then $|x_n - L| \leq \left|x_n - x_{n_{k_0}}\right| + |x_{n_k} - L| < \varepsilon$.

- If x_n is a Cauchy sequence, then it must converge.

  Proof: Then let x_{n_k} be a monotone subsequence. Since x_{n_k} is bounded, it must converge. Since x_n is Cauchy, it follows that x_n converges to the same limit.

- If x_n is a bounded sequence, then it has a convergent subsequence. [This is called the **Balzano-Weierstrass Theorem** – see page 57 of the text.] Modify the previous argument.