Least Squares Problems

$A = \begin{pmatrix} -a_1^T- \\ \vdots \\ -a_m^T- \end{pmatrix}$

$$\min_{x \in \mathbb{R}^n} \|Ax-b\|^2 = (a_1^T x - b_1)^2 + (a_2^T x - b_2)^2 + \cdots + (a_m^T x - b_m)^2$$

How to find minimizer

$$(Ax-b)^T(Ax-b)$$
$$= (x^TA^T - b^T)(Ax-b)$$
$$= x^TA^TAx - b^TAx - \boxed{x^TA^Tb} + b^Tb$$
$$= \underbrace{x^TA^TAx}_{Q} - \underbrace{2b^TAx}_{b^T} + b^Tb$$

$$x^* = Q^{-1}b = (A^TA)^{-1}A^Tb$$

Example Problems

12.2. Suppose we have a spring, and it stretches to lengths according to table

F	L
1	3
2	4
4	5

We want to fit a line to this data of form

$$L = a + bF$$

We want to determine a, b by Least Squares

~~Both~~ Setup:

Objective: minimize

$$\overbrace{(a + bF_1 - L_1)^2 + (a + bF_2 - L_2)^2 + (a + bF_3 - L_3)^2}^{f(a,b)}$$

Matrix

$$A = \begin{pmatrix} 1 & F_1 \\ 1 & F_2 \\ 1 & F_3 \end{pmatrix} \quad b = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}$$

$$\text{Solution} = (A^T A)^{-1} A b$$

Question: what if we want approximation of the form

$$L = a + bF + cF^2 + d\sin(F)$$

A: add columns $\begin{pmatrix} F_1^2 \\ F_2^2 \\ F_3^2 \end{pmatrix}$ and $\begin{pmatrix} \sin(F_1) \\ \sin(F_2) \\ \sin(F_3) \end{pmatrix}$

to matrix A.

Discrete Time-linear Systems

$x_{k+1} = ax_k + bu_k$ for $k = 0, \dots, n-1$

Suppose we know certain input, outputs

k	u_k	x_k	x_{k+1}
0	1	0	x_1^*
1	0	x_1^*	x_2^*
$\vdots$	0	x_2^*	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	0	x_{n-1}^*	x_n^*
n	$\cancel{0}$	x_n^*	

$$A = \begin{pmatrix} 0 & 1 \\ x_1^* & 0 \\ \vdots & \vdots \\ x_{n-1}^* & 0 \end{pmatrix} \qquad b = \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 0 & x_1^* & \cdots & x_{n-1}^* \\ 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_1^* & 0 \\ \vdots & \vdots \\ x_{n-1}^* & 0 \end{pmatrix} \qquad A^Tb =$$

$$= \begin{pmatrix} \sum_{k=1}^{n-1} x_k^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Min Norm Problem

$$\min \|x\|$$
$$\text{s.t. } Ax = b$$

Solution is $x^* = A^T(AA^T)^{-1}b$

~~Note~~: Proof: Suppose $Ax = b = Ax^*$
Then $A(x - x^*) = 0$

Note

$$(x-x^*)^T x^* = (x-x^*)^T A^T (AA^T)^{-1} b$$
$$= \underbrace{(A(x-x^*))^T}_{=0} (AA^T)^{-1} b$$

meaning $x - x^* \perp x^*$

Then for any x s.t. $Ax = b$,

$$\|x\|^2 = \|x - x^* + x^*\|^2$$
$$= \|x - x^*\|^2 + \|x^*\|^2$$
$$\geq \|x^*\|^2$$

Therefore x^* is a minimizer