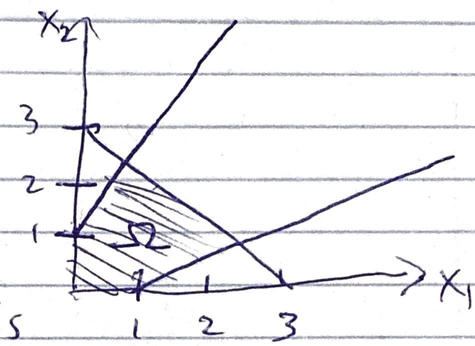
Linear program

Suppose we have the following LP

$$\min \; -x_1 - 2x_2$$

$$\text{s.t.} \quad -2x_1 + x_2 \le 1$$
$$x_1 - 2x_2 \le 1$$
$$x_1 + x_2 \le 3$$
$$x_1, x_2 \ge 0$$



In standard form, this is

$$\min \; -x_1 - 2x_2$$
$$\text{s.t.} \quad -2x_1 + x_2 + s_1 = 1$$
$$x_1 - 2x_2 + s_2 = 1$$
$$x_1 + x_2 + s_3 = 3$$
$$x_1, x_2, s_1, s_2, s_3$$

We can encode all the information inside a simplex Tableau

basic variables

nonbasic variables

x_1	x_2	s_1	s_2	s_3	b
-2	1	1	0	0	1
1	-2	0	1	0	1
1	1	0	0	1	3
-1	-2	0	0	0	0

These are current values of basic variables

this will hold the negative value of current objective.

Theseare from the objective function, $f(x) = -1x_1 - 2x_2$

This table is already canonical with respect to basis s_1, s_2, s_3. This means the columns for s_1, s_2, s_3 form the identity matrix, and so, if

x_1, x_2 are not in the basis (we set $x_1, x_2 = 0$)
It is very easy to solve the remaining system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

which tells us $s_1 = 1, s_2 = 1, s_3 = 3$.
Therefore if a tableaux is canonical with respect to a basis, the value of those basis variables is just the right hand side.

We would like to move x_1 or x_2 into the basis (i.e., make them nonzero) because the objective function $f(x) = -x_1 - 2x_2$, if we increase x_1 or x_2 we will decrease the value of the function.
Since the coefficient of x_2 is more negative, we will pick x_2.
Now we want to determine how much we can raise the value x_2.
Consider the first constraint

$$-2x_1 + x_2 + s_1 = 1$$

If ~~x_2~~ is in the basis and x_1, s_1 are out, ($x_1 = s_1 = 0$)
x_2 takes on a value of at most 1
For the second constraint,

$$x_1 - 2x_2 + s_2 = 1$$

we cannot satisfy this constraint if x_2 is in basis and $x_1, s_2 = 0$ since then $x_2 = -\frac{1}{2} \leq 0$,
so s_2 cannot leave basis, we can raise x_2 without bound and satisfy this constraint
For third constraint, $x_1 + x_2 + s_3 = 3$
If we have x_1, s_3 out of basis, x_2 in,
x_2 takes value at most 3

From each of this constraints, we are most restricted by the first.

In general, if we have variable x to move into basis with coefficient $a > 0$ and right hand side b, e.g.

$$ax + a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

the most we can raise x is $\frac{b}{a}$.

To determine which variable leaves the basis, find row where this ratio is smallest, and the basis variable leaving is the one with a 1 in this row.

x_1	x_2	s_1	s_2	s_3	b	Ratios of b to x_2 column
-2	1	1	0	0	1	1 ← smallest
1	-2	0	1	0	1	$-\frac{1}{2}$ ← negative, so ignore
1	1	0	0	1	3	3
-1	-2	0	0	0	0	

variable entering basis (x_2) Variable leaving basis (s_1)

To make change of basis, perform elimination so that x_2 column only has a 1 in the specified row and zero elsewhere

x_1	x_2	s_1	s_2	s_3	b	Ratios of b to x_1 column
-2	1	1	0	0	1	$-\frac{1}{2}$ negative ignore
-3	0	2	1	0	3	-1 negative ignore
3	0	-1	0	1	2	$\frac{2}{3}$ ← smallest
-5	0	2	0	0	2	

Enters basis (x_1) leaves basis (s_3)

Now we wish to introduce x_1 to the basis since it has the most negative cost coefficient. We remove s_3 from the basis

After elimination

x_1	x_2	s_1	s_2	s_3	b
0	1	1/3	0	2/3	7/3
0	0	1	1	1	5
1	0	$-\frac{1}{3}$	0	1/3	2/3
0	0	1/3	0	5/3	16/3

All cost coefficients nonnegative, we are at optimal solution.

The optimal value is in lower right, but negated.

$$-16/3$$

The solution is given by fixing nonbasis variables equal to zero

$$s_1 = s_3 = 0$$

and reading off the right hand column for the values of the basis variables

$$x_2 = 7/3 \quad s_2 = 5 \quad x_1 = 2/3$$

General Simplex Algorithm (Phase II algorithm)

1. Construct Simplex Tableau
2. Determine nonbasic variable to enter basis by finding variable with most negative cost coefficient.
3. Compute ratios of rightmost column to nonbasic variable column
4. Find row with minimal nonnegative ratio

   If no such ratio (all ratios negative or infinite (of the form a/0)) problem is unbounded
5. Perform Gaussian elimination on column so that this row has 1 in it and all other entries in column is zero
6. Repeat until all cost coefficients ≥ 0
7. Read off solution from final simplex tableau