Lagrange Multipliers

General form

$$\min f(x_1, \dots, x_n)$$
$$\text{s.t. } h_1(x_1, \dots, x_n) = 0$$
$$\vdots$$
$$h_m(x_1, \dots, x_n) = 0$$

To solve this problem, we consider the lagrangian

$L(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m)$

Note: I am using different notation from the book here. The lagrangian is denoted by lowercase L in the book

$$= f(x_1, \dots, x_n) + \lambda_1 h_1(x_1, \dots, x_n) + \dots + \lambda_m h_m(x_1, \dots, x_n)$$

Theorem (FONC): If x is a local min (or max), and regular, there exist $\lambda_1, \dots, \lambda_m$ such that

This means $\nabla h_1(x), \dots, \nabla h_m(x)$ are linearly independent.

$$\frac{\partial L}{\partial x_1}(x, \lambda) = 0$$
$$\vdots$$
$$\frac{\partial L}{\partial x_n}(x, \lambda) = 0$$

This means we have two sets of equations we can use to solve our problem:

$$\begin{cases} h_1(x) = 0 \\ \vdots \\ h_m(x) = 0 \end{cases} \qquad \begin{cases} \frac{\partial L}{\partial x_1}(x, \lambda) = 0 \\ \vdots \\ \frac{\partial L}{\partial x_n}(x, \lambda) = 0 \end{cases}$$

Example 20.15

min $x_1 x_2 - 2x_1$
s.t. $x_1^2 - x_2^2 = 0$

We construct Lagrangian

$$L(x,\lambda) = x_1 x_2 - 2x_1 + \lambda(x_1^2 - x_2^2) = 0$$

The conditions are

$$\frac{\partial L}{\partial x_1} = x_2 - 2 + 2\lambda x_1 = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = x_1 - 2\lambda x_2 = 0 \quad (2)$$

inaddition to

$$x_1^2 - x_2^2 = 0 \quad (3)$$

We solve this system of equations for x_1, x_2, λ.

From (3), we have $(x_1 + x_2)(x_1 - x_2) = 0$ so that $x_1 = x_2$ or $x_1 = -x_2$.

In the first case $(x_1 = x_2)$, the other 2 conditions are

$$x_1 - 2 + 2\lambda x_1 = 0$$

$$x_1 - 2\lambda x_1 = 0$$

and adding equations give

$$2x_1 - 2 = 0 \Rightarrow x_1 = 1$$

$$\Rightarrow x_2 = 1$$

$$\Rightarrow \lambda = \tfrac{1}{2}$$

$(x_1 = -x_2)$

In second case, the conditions are

$-x_1 - 2 + 2\lambda x_1 = 0$

$x_1 + 2\lambda x_1 = 0$

Subtraction equations gives

$-2x_1 - 2 = 0 \Rightarrow x_1 = -1$

$\Rightarrow x_2 = +1$

$\lambda = -\frac{1}{2}$

Two possible solutions $x = (-1, +1), \lambda = -\frac{1}{2}$

$x = (1, 1), \lambda = \frac{1}{2}$

To check if it is a minimizer, there are second order conditions.

We first define the tangent space:

At a point x, the tangent space $T(x)$, is defined

as $\{y : \nabla h_i(x)^T y = 0 \ \forall i\}$

i.e., the set of directions orthogonal to the gradient of each constraint function h_i.

We have

Theorem (SONC): If x is regular and a local min, ~~and~~ There exists λ satisfying FONC and for all $y \in T(x)$,

$$y^T(\nabla_x^2 L(x, \lambda))y \geq 0$$

Theorem (SOSC): If x, λ satisfies FONC and $y^T(\nabla_x^2 L(x, \lambda))y > 0$ for all $y \neq 0, y \in T(x)$, x is a strict local min.

Let's apply this to our problem.

The hessian of Lagrangian is

$$\nabla_x^2 L(x,\lambda) = \begin{pmatrix} 2\lambda & 1 \\ 1 & -2\lambda \end{pmatrix}$$

At each point we compute tangent space.

We have

$$\nabla h(x) = \begin{pmatrix} 2x_1 \\ -2x_2 \end{pmatrix}$$

$$\nabla h(1,1) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

The set of directions orthogonal is

$$T(1,1) = \{a\begin{pmatrix} 1 \\ 1 \end{pmatrix} : a \in \mathbb{R}\}$$

$$\nabla h(-1,+1) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

The set of orthogonal directions is

$$T(-1,+1) = \{a\begin{pmatrix} 1 \\ -1 \end{pmatrix} ; a \in \mathbb{R}\}$$

Now we check SONC

for $x = (-1,1)$, $\lambda = -\frac{1}{2}$,

$$\nabla^2 L(-1,1,-\tfrac{1}{2}) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

We consider. $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in T(-1,1)$

$$(1\ \ -1)\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1\ \ -1)\begin{pmatrix} -2 \\ 0 \end{pmatrix} = -2 < 0$$

Since this is negative, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ cannot be a local min by SONC.

For $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda = \frac{1}{2}$

$$\nabla^2 L(1,1,\tfrac{1}{2}) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Consider any direction in $T(1,1)$,

$\delta = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $a \neq 0$

$$\delta^T \nabla^2 L(1,1,\tfrac{1}{2})\delta = a(1\ \ 1)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= a^2 (1\ \ 1)\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2a^2 > 0$$

Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satisfies SOSC and is a local min.