~~min f x~~

Setup $\quad \min f(x)$

$$\left.\begin{aligned} &g_1(x) \le 0 \\ &\vdots \\ &g_m(x) \le 0 \end{aligned}\right\} \text{Inequality constraints}$$

$$\left.\begin{aligned} &h_1(x) = 0 \\ &\vdots \\ &h_p(x) = 0 \end{aligned}\right\} \leftarrow \text{Equality constraints}$$

Lagrangian

$$L(x, \lambda, \mu) = f(x) + \lambda_1 h_1(x) + \dots + \lambda_p h_p(x) + \mu_1 h_1(x) + \dots + \mu_m g_m(x)$$

Ex: $\quad \min x_1^2 + x_2^2$

s.t. $\quad x_1 \le 3 \leftarrow$ 1 inequality

$x_1 x_2 = 1$

$x_1^2 = 2$ ← 2 equalities

Lagrangian is

$$L(x, \lambda, \mu) = x_1^2 + x_2^2 + \lambda_1(x_1 x_2 - 1) + \lambda_2(x_1^2 - 2) + \mu_1(x_1 - 3)$$

Note: If $\mu \geq 0$,

For x feasible

$$L(x,\lambda,\mu) = f(x) + \sum_i \lambda_i h_i(x) + \sum_j \mu_j g_j(x)$$

$$\leq f(x) \quad \text{since } h(x) = 0, \; g(x) \leq 0$$

Therefore

$$\underbrace{\min_{x} L(x,\lambda,\mu)}_{} \leq \min_{\substack{x \\ x \text{ feasible}}} f(x)$$

$$= \boxed{F(\lambda,\mu)}$$

~~Weak~~

~~λ Liner~~

This property is called Weak Duality

$$\begin{aligned}
\min \quad & 3x_1 + 4x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \cancel{\geq} = 1 \\
& 3x_1 + 2x_2 \cancel{\geq} = 2 \\
& x_1, x_2 \geq 0
\end{aligned}$$

$$\begin{aligned}
L(x,\lambda,\mu) = 3x_1 + 4x_2 &+ \lambda_1(2x_1 + x_2 - 1) \\
&+ \lambda_2(3x_1 + 2x_2 - 2) \\
&- \mu_1 x_1 \\
&- \mu_2 x_2
\end{aligned}$$

$$= -\lambda_1 - \lambda_2 + \underbrace{(3 + 2\lambda_1 + 3\lambda_2 - \mu_1)}x_1 + \underbrace{(4 + \lambda_1 + 2\lambda_2 - \mu_2)}_{\text{if } \neq 0}x_2$$

$$\min_x L = -\infty$$

Dual Problem: ~~#&sum~~ :

$$\max \; -\lambda_1 - \lambda_2$$

$$\text{s.t. } 3 + 2\lambda_1 + 3\lambda_2 - \mu_1 = 0$$

$$4 + \lambda_1 + 2\lambda_2 - \mu_2 = 0$$

$$\mu_1, \mu_2 \geq 0$$

$$\min \; x^T A x + 2b^T x$$

$$\text{s.t. } x^T x \leq 1$$

Lagrangian

$$x^T A x + 2b^T x + \lambda(x^T x - 1)$$

$$L(x, \lambda) = x^T(A + \lambda I)x + 2b^T x - \lambda$$

$$\nabla_x L = 2(A + \lambda I)x + 2b = 0$$

$$x = -(A + \lambda I)^{-1} b$$

$$\max_{\lambda \geq 0}$$

Dual Function

$$F(\lambda) = b^T(A + \lambda I)^{-1} b \; ~~2b^T~~$$