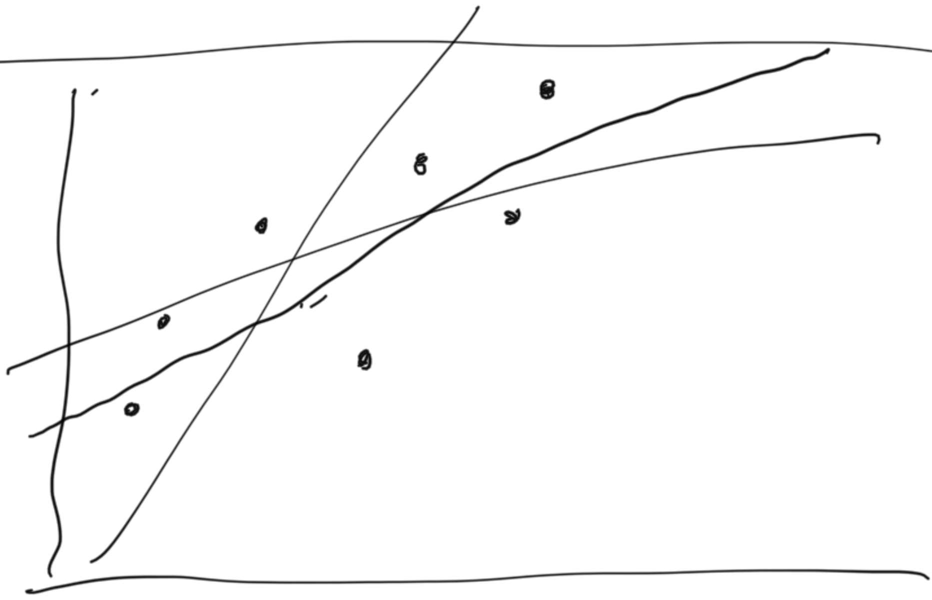
Discussion #7

1. Least Squares Regression
2. Linear Programs (LPs)



$$\boxed{\min_x \frac{1}{2} \|Ax-b\|^2}$$

$$f(t) = a_1 + a_2 t + a_3 t^2$$

t	0	1	3	4
y	0	2	1	3

Want to find a_1, a_2, a_3 such that

$$f(0) \approx 0$$

$$f(1) \approx 2$$

$$f(3) \approx 1$$

$$f(4) \approx 3$$

$$a_1 + a_2(0) + a_3(0)^2 \approx 0$$
$$a_1 + a_2(1) + a_3(1)^2 \approx 2$$
$$a_1 + a_2(3) + a_3(3)^2 \approx 1$$
$$a_1 + a_2(4) + a_3(4)^2 \approx 3$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{x} \approx \underbrace{\begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}}_{b}$$

$$\min \frac{1}{2} \|Ax - b\|_2^2$$

$$x^* = (A^T A)^{-1} A^T b$$

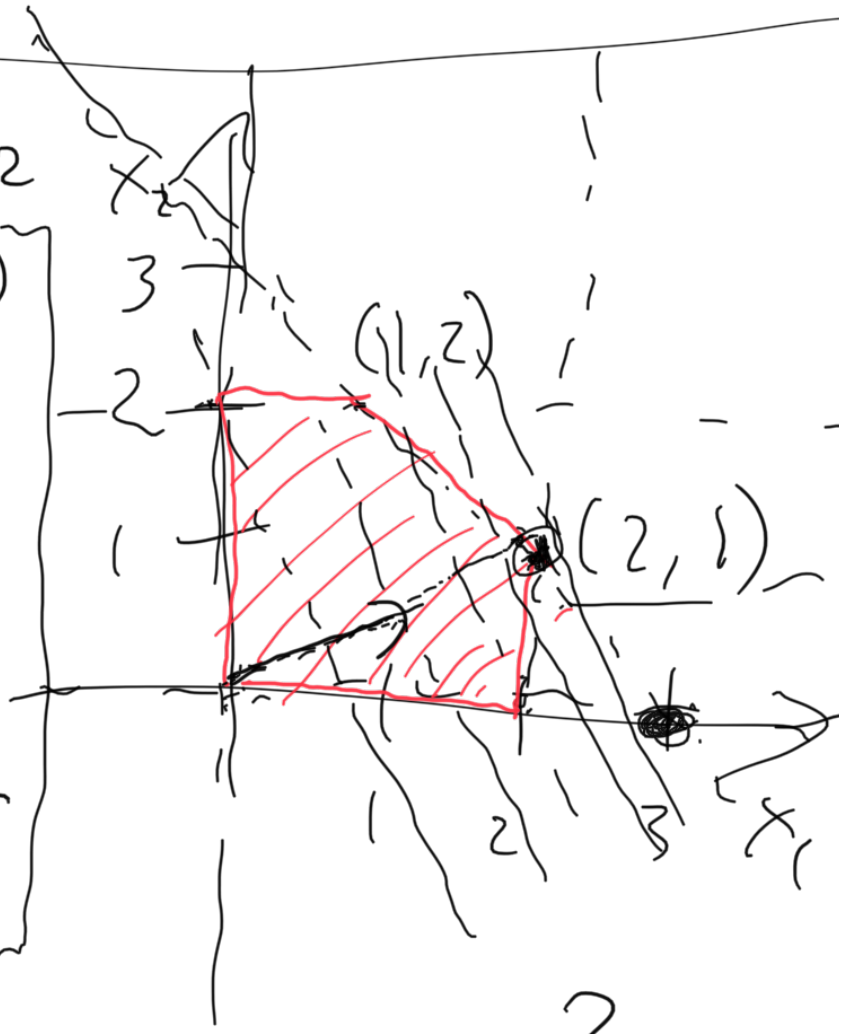
$$= \left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

Linear programming (LPs)

$$\min \; c^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

$$\max \; 2x_1 + x_2$$

s.t.
$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_1 \leq 2$$
$$x_2 \leq 2$$
$$x_1 + x_2 \leq 3$$



Where is the optimum occur?

1) Why can't this happen on interior?

FONC: $\nabla f(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \neq 0$

2) Where does the optimum occur?

$[2,1]$

Fact: Optima of LPs always occur at vertices of feasible polyhedron.

Possible Algorithm:

Enumerate all vertices of feasible polyhedron, check which vertex achieves optimal value.

Problem: There are potentially exponentially many vertices

For above problem, to find vertices of polyhedron, since we're in $\mathbb{R}^2$, we pick 2 inequalities to satisfy at equality, solve linear system.

Optimal: Choose $x_1 = 2$

$x_1 + x_2 = 3$

$\Rightarrow x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

possible vertices is $\binom{5}{2}$

In general, if we have m inequalities in $\mathbb{R}^n$, have $\binom{m}{n}$ vertices.

Note: not all choices lead to vertex,

e.g., choose $x_1 \geq 0$, $x_1 \leq 2$

$\Rightarrow x_1 = 0$, $x_1 = 2$, inconsistent

e.g., $x_1 + x_2 \leq 3$, $x_1 \geq 0$

$\Rightarrow x_1 = 0$, $x_2 = 3$ which is not feasible.

Standard Form LPs

minimization

$$\min \ c^T x$$
$$\text{s.t. } Ax = b \leftarrow \text{equality constraints}$$
$$x \geq 0 \leftarrow \text{nonnegativity constraints.}$$

How to convert to standard form?

1. Change max to min

1. Change ...
by negating objective

2. Change inequality constraints to equality constraints with slack/surplus variables, e.g.

$x_1 \le 2, \Rightarrow x_1 + s_1 = 2, \; s_1 \ge 0$

$x_1 \ge 2 \Rightarrow x_1 - s_1 = 2, \; s_1 \ge 0$

3. If x does not have a nonnegativity constraints (x is free)

write $x = x^+ - x^-, \; x^+, x^- \ge 0$

$$\begin{array}{ll} \min x_1 + x_2 & \min x_1^+ - x_1^- + x_2 \\ \text{s.t. } x_1 + x_2 = 1 \Rightarrow & \text{s.t. } x_1^+ - x_1^- + x_2 = 1 \\ x_2 \ge 0 & x_1^+, x_1^-, x_2 \ge 0 \\ x_1 \in \mathbb{R} & \end{array}$$

Other constraints?

How would convert $|x| < 1$

How would ...

to standard form?

$x \le 1$ and $x \ge -1$

Convert to equality constraints with slack/surplus variables

$$\boxed{\begin{gathered} x + s_1 = 1, \quad x - s_2 = -1 \\ s_1, s_2 \ge 0 \end{gathered}}$$

$\boxed{\max 2x_1 + x_2}$

s.t. $x_1 \ge 0$

$x_2 \ge 0$

$x_1 \le 2$

$x_2 \le 2$

$x_1 + x_2 \le 3$

In standard form:

min $-2x_1 - x_2$

s.t. $x_1 + s_1 = 2$

$x_2 + s_2 = 2$

$x_1 + x_2 + s_3 = 3$

$\boxed{x_1, x_2, s_1, s_2, s_3 \geq 0}$

Solutions are $(x, s) \in \mathbb{R}^5$

What are vertices of this polyhedron?

Since we are in $\mathbb{R}^5$, vertices correspond to intersection of 5 hyper planes.

Already have 3 equations, Need 2 more, Can get them by choosing 2 of x_1, x_2, s_1, s_2, s_3 $= 0$. $\binom{5}{2}$ different candidates

Choose $s_1, s_3 = 0$

$x_1 \qquad = 2$

$x_2 + s_2 = 2$

$$x_1 + x_2 = 3$$

$$\Rightarrow \boxed{\boxed{x_1 = 2,\ x_2 = 1,}\ s_2 = 1}$$

These solutions found by choosing certain variables equal to 0 and solving the remaining system. These are called basic solutions.