Discussion #7

1. Least Squares Regression
2. Linear Programs (LPs)

\[
\min_{x} \frac{1}{2} \|Ax - b\|^2 \\
f(t) = a_1 + a_2 t + a_3 t^2
\]

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Want to find \( a_1, a_2, a_3 \) such that

\[
\begin{align*}
f(0) & \approx 0 \\
f(1) & \approx 2 \\
f(\frac{1}{3}) & \approx 1
\end{align*}
\]
\[ f(4) \approx 3 \]
\[ a_1 + a_2(0) + a_3(0)^2 \approx 0 \]
\[ a_1 + a_2(1) + a_3(1)^2 \approx 2 \]
\[ a_1 + a_2(3) + a_3(3)^2 \approx 1 \]
\[ a_1 + a_2(4) + a_3(4)^2 \approx 3 \]

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 3 & 9 \\
1 & 4 & 16 \\
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
2 \\
1 \\
3 \\
\end{pmatrix}
\]

\[
\begin{align*}
A & = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 3 & 4 \\
0 & 1 & 9 & 16 \\
\end{pmatrix} \\
A^T & = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
3 & 3 & 9 \\
14 & 16 & 16 \\
\end{pmatrix}
\end{align*}
\]

\[
\min \frac{1}{2} \| A \mathbf{x} - b \|_2^2
\]

\[
\mathbf{x} = (A^T A)^{-1} A^T b
\]
Linear programming (LPs)

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_1 \leq 2 \\
& \quad x_2 \leq 2 \\
& \quad x_1 + x_2 \leq 3.
\end{align*}
\]

Where is the optimum occur?

1) Why can't this happen on interior?

FONC: \( \nabla f(x) = (1, 2) \neq 0 \)

2) Where does the optimum occur?
Fact: Optima of LPs always occur at vertices of feasible polyhedron.

Possible Algorithm:
Enumerate all vertices of feasible polyhedron, check which vertex achieves optimal value.

Problem: There are potentially exponentially many vertices

For above problem, to find vertices of polyhedron, since we're in $\mathbb{R}^2$, we pick 2 inqualities to satisfy at equality, solve linear system.

Optimal: Choose $X_1 = 2$

\[ X_1 + X_2 = 3 \]

\[ \Rightarrow x = \left( \frac{1}{2}, 1 \right) \]
The number of possible vertices is \((\binom{n}{2})\).

In general, if we have \(m\) inequalities in \(\mathbb{R}^n\), we have \((\binom{m}{n})\) vertices.

Note: not all choices lead to vertices.

E.g., choose \(x_1 \geq 0, x_1 \leq 2\)

\(\Rightarrow\) \(x_1 = 0, x_1 = 2\), inconsistent

E.g., \(x_1 + x_2 \leq 3, x_i \geq 0\)

\(\Rightarrow\) \(x_1 = 0, x_2 = 3\), which is not feasible.

---

**minimization standard form LPs**

\[
\min \quad c^T x
\]

s.t. \(A x = b\) \(\Leftarrow\) equality constraints

\(x \geq 0\) \(\Leftarrow\) nonnegativity constraints

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**How to convert to standard form?**

1. Change \(\max\) to \(\min\)
1. Change objective by negating objective.

2. Change inequality constraints to equality constraints with slack/surplus variables, e.g.,
\[
\begin{align*}
X_1 \leq 2 & \Rightarrow X_1 + S_1 = 2, S_1 \geq 0 \\
X_1 \geq 2 & \Rightarrow X_1 - S_1 = 2, S_1 \geq 0
\end{align*}
\]

3. If \( x \) does not have a nonnegativity constraint (\( x \) is free), write \( x = x^+ - x^- \), \( x^+, x^- \geq 0 \)

\[
\begin{align*}
&\min x_1 + x_2 \\
&\min x^+_1 - x^-_1 + x_2 \\
&\text{s.t. } x_1 + x_2 = 1 \Rightarrow \text{s.t. } x^+_1 - x^-_1 + x_2 = 1 \\
&x_2 \geq 0 \\
&x_1 \in \mathbb{R}
\end{align*}
\]

\[x^+_1, x^-_1, x_2 \geq 0\]

---

Other constraints?

If \( \ldots \) convert \( |x| < 1 \)
How would you convert $X \leq 1$ and $X \geq -1$ to standard form?

Converting to equality constraints with slack/surplus variables:

$$X + s_1 = 1, \quad X - s_2 = -1$$

$$s_1, s_2 \geq 0$$

Maximize $2x_1 + x_2$

Subject to:

- $x_1 \geq 0$
- $x_2 \geq 0$
- $x_1 \leq 2$
- $x_2 \leq 2$
- $x_1 + x_2 \leq 3$

In standard form:

Minimize $-2x_1 - x_2$

Subject to:

- $x_1 + s_1 = 2$
- $x_2 + s_2 = -2$
- $s_1, s_2 \geq 0$
\[
\begin{align*}
    x_2 + x_1 &= 2 \\
    x_1 + x_2 + s_3 &= 3 \\
    x_1, x_2, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

Solutions are \((x, s) \in \mathbb{R}^5\).

What are vertices of this polyhedron?

Since we are in \(\mathbb{R}^5\), vertices correspond to intersection of 5 hyperplanes.

Already have 3 equations, need 2 more, can get them by choosing 2 of \(x_1, x_2, s_1, s_2, s_3\) = 0. (5) different candidates.

Choose \(s_1, s_3 = 0\)

\[
\begin{align*}
    x_1 &= 2 \\
    x_2 + s &\geq 2
\end{align*}
\]
These solutions found by choosing certain variables equal to 0 and solving the remaining system. These are called basic solutions.